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RELATIVISTIC IRREVERSIBLE THERMODYNAMICS

by

DONALD AITKEN

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Relativistic Irreversible Thermodynamics" submitted by Donald Aitken in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

Relativistic treatments of irreversible processes in a simple fluid have been given by C. Eckart and by Landau and Lifshitz. Superficially, the two formulations bear little resemblance to each other. The basis of the two theories has been explored by kinetic-theoretical methods and it is understood that the difference between them is due simply to a different choice of rest frame.

In the present thesis, we have attempted the construction of a general phenomenological theory which is not tied at the outset to a particular choice of rest frame, and which will include the Landau-Lifshitz and Eckart formalisms as special cases. A general formulation of this type leads to a number of aesthetically satisfying results, for instance the amalgamation of Fick's law of particle diffusion and Fourier's law of heat conduction into a single law, in conformity with the relativistic equivalence of mass and energy. The extension to fluid mixtures, and the incorporation of electromagnetic effects (of interest in plasma theory) are particularly simple.

The complete flexibility in the choice of rest frame also brings with it certain practical advantages, since a judicious choice will often simplify the mathematics. We have illustrated this (a) by giving new and simple proofs of the theorems of Tolman and Klein on the variation of temperature and thermal potential in a fluid in thermal equilibrium in a static gravitational field; (b) by integrating the equations for one-dimensional steady flow in the presence of viscosity and heat conduction to obtain an estimate for the thickness of a relativistic plane shock layer.

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CHAPTER I

INTRODUCTION.

The body of the thesis will be concerned with construction of a relativistically invariant theory of the thermodynamics of irreversible processes. The classical theory of irreversible processes (and also the two relativistic theories of Eckart [1] and Landau-Lifshitz [3]) consists of a linear model of the irreversible processes based on the theory of reversible processes. Such a theory, of course, can be expected to apply only to processes near to equilibrium.

Before we embark on the construction of our theory (also a linear approximation), it may be helpful to review the classical theory of reversible and irreversible processes. While so doing we shall pinpoint the assumptions which brand it as a linear approximation.

Section 1: The Theory of Equilibrium (Reversible) Thermodynamics.

In the state of equilibrium a simple fluid may be described by three parameters:

1.1.1
(i) u = energy per gram
(ii) v = volume per gram
(iii) s = entropy per gram

The entropy may be expressed in terms of u and v by an equation of state

1.1.2 $$s = s(u,v) \quad .$$

We may define T and P by

1.1.3 $$Tds = du + Pdv$$

thus making

1.1.4 $$T = \left[\left(\frac{\partial s}{\partial u}\right)_v\right]^{-1} \; ; \quad P = \left(\frac{\partial s}{\partial v}\right)_u T \; .$$

If we let dq (not a perfect differential) be the amount of energy transferred from the surroundings to one gram of the fluid, we may postulate

1.1.5 $$Tds = dq \; .$$

It is easy to see that P is the mechanical pressure, since

1.1.6 $$dq = \text{increase in energy of 1 gm of fluid} + \text{work done by 1 gm of fluid.}$$

Section 2: The Classical Theory of Non-equilibrium (Irreversible) Processes.

We begin by assuming (Assumption 1) that an equation of state

1.2.1 $$s = s(u,v)$$

still exists during the irreversible process. Next we assume that s is differentiable (Assumption 2) so that we may write

1.2.2 $$Tds = du + Pdv \; .$$

If we put

1.2.3
$$\vec{q} = \text{energy flux vector}$$
$$t_{ij} = \text{mechanical stress tensor}$$

we may express the conservation of energy as

1.2.4 $$\frac{du}{dt} + v \operatorname{div} \vec{q} + v\, t_{ij} \frac{\partial v_i}{\partial x_j} = 0 \quad .$$

We may define the viscous stress tensor τ^{ij} by

1.2.5 $$\tau^{ij} = t^{ij} - P\delta_{ij} \quad .$$

The equation of continuity

1.2.6 $$\frac{d\rho}{dt} = -\rho \operatorname{div} \vec{v} \quad ; \quad \rho = \frac{1}{v}$$

enables us to write 1.2.4 as

1.2.7 $$\rho \frac{ds}{dt} + \operatorname{div} \left(\frac{\vec{q}}{T}\right) = \frac{-\tau_{ij}}{T} \frac{\partial v_i}{\partial x_j} + \vec{q} \cdot \nabla \left(\frac{1}{T}\right) \quad .$$

Observe that for an isolated volume v

1.2.8 $$\frac{d}{dt} \iiint_v \rho A \, dv = \iiint_v \frac{dA}{dt} \rho \, dv + \iiint_v A \frac{d}{dt} (\rho dv)$$

$$= \iiint_v \frac{dA}{dt} \rho dv \quad .$$

The second law of thermodynamics asserts that the total entropy of an isolated system is non-decreasing. Thus

1.2.9 $$\frac{d}{dt} \iiint_V \rho s \, dv \geq 0 \quad .$$

In view of 1.2.8, 1.2.7 and the fact that $\vec{q} = 0$ on the boundary of an isolated region, we may put

1.2.10 $$\frac{d}{dt} \iiint \rho s \, dv = \iiint_V \left\{ - \tau_{ij} \frac{\partial v_i}{\partial x_j} + \vec{q} \cdot \nabla \frac{1}{T} \right\} dv \geq 0 \quad .$$

To put the entropy production on a local basis we will suppose that entropy can "flow". Thus

1.2.11 $$\text{Entropy Production} = \text{Entropy Increase} + \text{Entropy Export}$$
$$= \frac{d}{dt} \iiint_V \rho s \, dv + \iint_V \vec{s} \cdot \vec{n} \, dA$$

where $\vec{s}$ is the vector representing the "flow" of entropy. The rate of entropy production (σ) per unit volume is then by 1.2.11

1.2.12 $$\sigma = \rho \frac{ds}{dt} + \operatorname{div} \vec{s} \quad .$$

Next, to identify the rate of entropy production at a point, we assume (Assumption 3) that

1.2.13 $$\vec{s} = \frac{\vec{q}}{T} \quad .$$

This identifies the local entropy production, for according to equations 1.2.7, 1.2.13

1.2.14 $$\sigma = -\frac{1}{T}\tau^{ij}\frac{\partial v_i}{\partial x_j} + \vec{q}.\nabla\left(\frac{1}{T}\right) .$$

If we now assume that $\sigma \geq 0$ (Assumption 4) ; that τ^{ij} and $\vec{q}$ are linear functions of $\frac{\partial v_i}{\partial x_j}$ and $\nabla\left(\frac{1}{T}\right)$ (Assumption 5) we may deduce Fourier's law

1.2.15 $$\vec{q} = K.\nabla\left(\frac{1}{T}\right) \qquad K \geq 0$$

and the form of the viscous stress tensor

1.2.16 $$\tau^{ij} = -a\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - b\frac{\partial v_k}{\partial x_k}\delta_{ij}$$

$$a, b \geq 0 .$$

<u>Section 3: Discussion of the Assumptions of Classical Irreversible Thermodynamics</u>.

Let us set down the assumptions made in Section 2.

1. An equation of state $s = s(u,v)$ exists under non-equilibrium conditions.

2. The relation $Tds = du + Pdv$ holds under non-equilibrium conditions.

3. A linear relation exists between $\vec{s}$ and $\vec{q}$: $\vec{s} = \frac{\vec{q}}{T}$.

4. A linear relation holds between the "fluxes" $\vec{q}$, τ^{ij} and the "forces" $\nabla(\frac{1}{T})$, $\frac{\partial v_i}{\partial x_j}$.

5. The rate of entropy production is positive.

The assumptions of linearity in 3, 4, 5 suggest that the theory is only applicable near equilibrium. If we appeal to kinetic theory (see Prigogine [11]) we find that, for gases, only (5) can be justified for arbitrary deviations from equilibrium (it is Boltzmann's H Theorem). The assumptions 1 - 4 can only be justified for near-equilibrium conditions. It appears then, that the method of section 2 supplies only a linear model for what is most likely a non-linear physical situation.

It is worth noting that assumption 3 may be replaced by the weaker assumption

1.3.1 $$\operatorname{div} \vec{s} = \operatorname{div} \frac{\vec{q}}{T} .$$

In any case, the energy flux $\vec{q}$ is only determined to within curl $\vec{A}$, when $\vec{A}$ is any vector whose curl vanishes on the boundary of an isolated system. After all, we can only ascribe unique meaning to heat flow through a closed surface, i.e. to

1.3.2 $$\iint_S \vec{q} \cdot \vec{n} \, dA \qquad (S \text{ closed}) .$$

If

1.3.3 $$\vec{q}_* = \vec{q} + \text{curl } \vec{A}$$

we have

1.3.4 $$\iint_S \vec{q}_* \cdot \vec{n} \, dA = \iint \vec{q} \cdot \vec{n} \, dA \quad (S \text{ closed}) \quad .$$

Hence, from the point of view of measurement, $\vec{q}_*$ is just as good a candidate as $\vec{q}$ for use in the equation of conservation of energy. If we do use it, we get Fourier's law in the form

1.3.5 $$\vec{q}_* = K \cdot \nabla \left(\frac{1}{T}\right)$$

which is different from the result

$$\vec{q} = K \nabla \left(\frac{1}{T}\right) \quad .$$

However, Fourier's law for a closed surface S is

$$\iint_S \vec{q} \cdot \vec{n} \, dA = \iint_S K \nabla \left(\frac{1}{T}\right) \cdot \vec{n} \, dA \quad .$$

Since

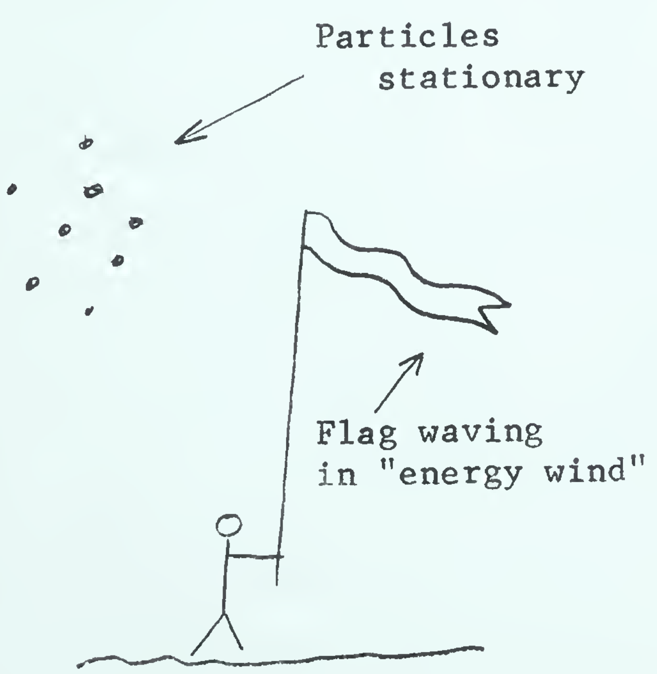
$$\iint_S \vec{q}_* \cdot \vec{n} \, dA = \iint_S \vec{q} \cdot \vec{n} \, dA$$

the result for closed surfaces (the only one with unique physical meaning) is the same for either choice of $\vec{q}$.

Section 4: Relativistic Formulation.

The points made in section 3 do not preclude construction of a regorous, self contained mathematical model of irreversible thermodynamics based on assumptions 1 - 5. Such a model of course, would have its range of applicability restricted to near equilibrium conditions. It would be analogous to Newtonian Mechanics - an elegant model of the real world whose applicability is restricted to "classical" conditions. In fact, such models have been constructed, using the frame work of relativity, by Eckart [1] in 1940 and later by Landau and Lifshitz [3].

The two theories suffer from a basic lack of invariance with respect to the rest frame of the observer. Eckart describes the behaviour of the fluid as seen by an observer who sees no net particle flux. Landau-Lifshitz developed their theory from the point of view of an observer who sees no net energy flux. If we think of Eckart and Landau as equipped with small flags, standing in their respective reference frames, the situation is:



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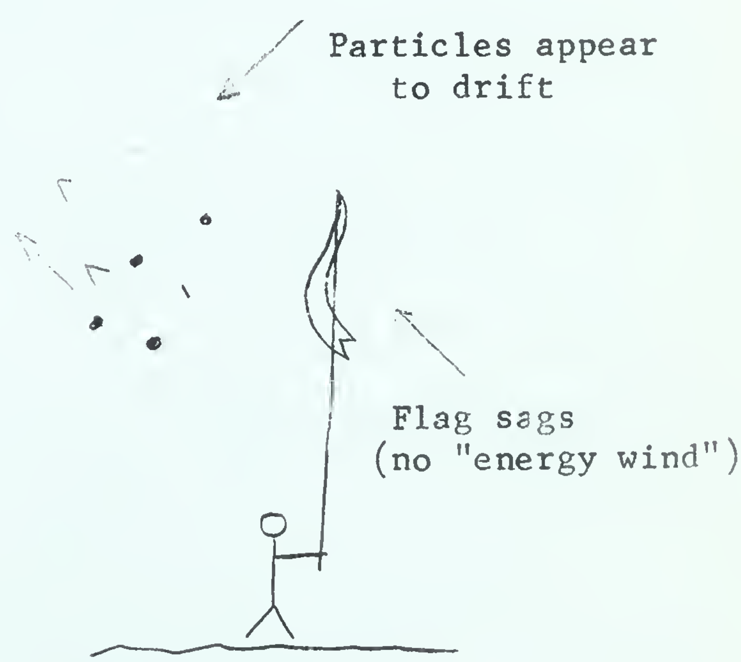

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Eckart's flag will flap in the energy "wind" but he will see the particles as stationary on the average; Landau's flag will be limp on its staff but he will see a net drift of particles past him.

The results obtained by the two men differ in form, but, as we shall see, are consistent. Landau describes the observed particle flux as proportional to (-) the gradient of thermodynamic potential; Eckart describes the observed energy flow as proportional to (+) the gradient of thermodynamic potential. The form of the mechanical stress tensor deduced by the two is identical. Since the velocities on which the stress tensor depends are different for the two observers, the result is in fact at variance. We will show that the difference is negligible.

In the process of demonstrating the equivalence of the two theories, we have constructed a third theory which is not tied at the outset to the rest frame of any particular observer. By suitable specialization, we may obtain Eckart's theory or Landau's theory as a special case.

CHAPTER II

A GENERALIZED FORMALISM FOR THE IRREVERSIBLE THERMODYNAMICS OF A SIMPLE FLUID

Section 1.

In this chapter we will consider the thermodynamics of a simple fluid - one composed of particles of only one kind. It will be described macroscopically by a symmetric tensor $T^{\mu\nu}$ and two vectors M^{μ} and S^{μ}, which we shall take as given a priori. We will from the outset assume the conservation equations

2.1.1 $$T^{\mu\nu}{}_{,\nu} = 0$$

2.1.2 $$M^{\mu}{}_{,\mu} = 0$$

and the principle of increase of entropy

2.1.3 $$S^{\mu}{}_{,\mu} \geq 0 \ .$$

We will use the framework of general relativity, and utilize a metric with signature + 2. As for notation, a comma stands for covariant differentiation, Greek indices run from 1 to 4, and Latin indices run from 1 to 3. Although it is irrelevant to a formal phenomenological description of the fluid, it may be helpful to explain the meaning of these quantities from a microscopic point of view. To do this consider an observer 0, moving with 4-velocity U^{α}, and carrying a small imaginary box of dimensions $dx : dy : dz$ whose faces are perpendicular to the $x : y : z$ axes of his local (Cartesian) coordinate system. This observer will denote quantities measured in his local coordinate system by a star (*) ; in particular $U^{\mu} \overset{*}{=} (0,0,0,1)$. For

simplicity we will use units wherein $c = 1$.

A. M^μ has the property that $|M_\mu N^\mu| d\Sigma$ is the number of world lines of fluid particles which intersect a 3-dimensional subspace with volume $d\Sigma$, multiplied by the rest mass of one particle (N^μ is the normal to $d\Sigma$) . The meaning of this statement for O is easy to see. Taking the subspace $d\Sigma$ to be his box, $N^\mu \overset{*}{=} (0,0,0,1)$, $d\Sigma \overset{*}{=} dV = dx\, dy\, dz$ and $|M_\mu N^\mu| d\Sigma \overset{*}{=} M^4 dV$. Since his box is part of the hypersurface t = constant, $M^4 dV$ is the rest mass of the particles in his box at time t . On the other hand, taking the subspace $d\Sigma$ to be the history of the face of the box perpendicular to the x axis during time dt , we have

$$N^\mu \overset{*}{=} (1,0,0,0) \quad , \quad d\Sigma \overset{*}{=} dy\, dz\, dt \quad ,$$

and

$$|M_\mu N^\mu| d\Sigma \overset{*}{=} M^1\, dA\, dt \quad .$$

Since our subspace is now a portion of the hypersurface x = constant, M^1 is the total rest mass of the particles crossing unit area perpendicular to the x axis in unit time. Similar results hold for M^2 and M^3 . To summarize: M^1 , M^2 , M^3 are the total rest masses of the particles crossing unit areas perpendicular to the x axis, y axis, z axis in unit time; M^4 is the density of rest mass.

The significance of the conservation equation $M^\mu{}_{,\mu} = 0$ is now easy to see. Considering a small region R at rest in O's local coordinate system and the fact that in local cartesian systems covariant differentiation reduces to ordinary partial differentiation,

2.1.4 $$\iiint_R \frac{\partial M^4}{\partial x^4}\, dV \overset{*}{=} - \iiint_R \frac{\partial M^i}{\partial x^i}\, dV .$$

Using Gauss' theorem on the R.H.S.

2.1.5 $$\frac{d}{dt} \iiint_R M^4\, dV \overset{*}{=} - \iint_R M^i n_i\, dA .$$

2.1.5 states that the number of particles is conserved, i.e. the only change in the number of particles inside the volume is due to a flux of particles across the surface of the region.

B. $T^{\mu\nu}$ has the following properties:

1. $T^{\mu\nu} U_\mu U_\nu\, dV \overset{*}{=} T^{44}\, dV$ is the total energy inside a region of volume dV.

2. $T^{\mu\nu} (\delta^i_\mu + U^i U_\mu)\, U_\nu dV \overset{*}{=} T^{i4} dV$ is the i^{th} component of the total momentum inside a region of volume dV.

3. $T^{\mu\nu} (\delta^i_\mu + U^i U_\mu)(\delta^j_\nu + U^j U_\nu) dA \overset{*}{=} T^{ij} dA$ is the i^{th} component of the force exerted on a face of the box perpendicular to the x^j axis by the material corresponding to smaller values of x^j. (dA is the area of the face.)

The interpretation of the conservation equation $T^{\mu\nu}_{\ ,\nu} = 0$ is now easy to see. Consider the total force on the material in the observers box. The forces tending to push it in the direction of positive x^i are

$$T_{i1}\, dx_2\, dx_3 + T_{i2}\, dx_1\, dx_3 + T_{i3}\, dx_1\, dx_2 \quad .$$

The forces tending to push it in the direction of negative x^i are

$$- \left(T_{i1} + \frac{\partial T_{i1}}{\partial x^1} dx_1\right) dx_2\, dx_3 \; - \; \left(T_{i2} + \frac{\partial T_{i2}}{\partial x^2} dx_2\right) dx_2\, dx_3$$

$$- \left(T_{i3} + \frac{\partial T_{i3}}{\partial x^3} dx_3\right) dx_1\, dx_2 \quad .$$

Thus the total force pushing the material in the box in the direction of positive x^i is

$$- \frac{\partial T_{ij}}{\partial x^j} dV \quad .$$

By virtue of $\frac{\partial T_{i\mu}}{\partial x_\mu} \overset{*}{=} 0$

2.1.6 $$\frac{\partial T_{i4}}{\partial t} dV \overset{*}{=} - \frac{\partial T_{ij}}{\partial x^j} dV \quad .$$

This, since $T_{i4}\, dV$ is the i^{th} component of the total momentum in the box, just means that the rate of change of momentum is equal to the force applied.

Consider next the flow of energy. In relativity, energy μ has mass $\frac{\mu}{c^2}$, an energy flow $\vec{\mu}$ has momentum $\frac{\vec{\mu}}{c^2}$, and, conversely, momentum $\vec{m}$ implies an energy flow $\vec{m}\, c^2$. Consider now a small region R fixed with respect to the observer O . By virtue of $T^{4\mu}_{\ \ ,\mu} = 0$

$$\iiint_R \frac{\partial T^{4\mu}}{\partial x^\mu} dV \overset{*}{=} 0$$

or, using Gauss' theorem

2.1.7 $$\frac{d}{dt} \iiint_R T^{44}\, dV \overset{*}{=} - \iint_R T^{4i} n_i \, dA \quad .$$

The L.H.S. is the time rate of change of energy, the R.H.S. (with $c = 1$) is the influx of energy across the boundary.

To summarise, the conservation equations $T^{\mu\nu}_{\ ,\nu} = 0$ are expressions of the conservation of energy $(T^{4\nu}_{\ ,\nu} = 0)$ and the conservation of momentum $(T^{i\nu}_{\ ,\nu} = 0)$.

C. S^{μ} is the entropy flux vector. Since we are considering a fluid with irreversible changes taking place within it, entropy is being created within it. Thus, the entropy outflux from a region, plus the entropy increase within the region, must be positive. Taking S^{i} as the entropy flux vector and S^{4} as the entropy density

2.1.8 $$\iint_R S^{i} n_i \, dA + \frac{d}{dt} \iiint_R S^{4}\, dV \geq 0$$

or using Gauss theorem for the arbitrary (small) region R

2.1.9 $$S^{\mu}_{\ ,\mu} \geq 0 \quad .$$

D. It is worth noting that the timelike eigenvector of $T^{\mu\nu}$ has great physical significance. Define E^{μ} by

2.1.10 $$T^{\mu\nu} E_{\nu} = - \varphi E^{\mu} \quad ; \quad E^{\mu} E_{\mu} = -1 \quad .$$

In a coordinate system where

2.1.11 $$E^{\mu} \overset{*}{=} [0,0,0,1] \; ,$$

we have

$$T^{\mu 4} \overset{*}{=} -\varphi E^{\mu} .$$

Thus E^{μ} is a unit vector parallel to the momentum.

Section 2: Reversible Thermodynamics.

Every fluid (which is not externally constrained to do otherwise) tends in time to achieve a state of equilibrium. This state is characterised by 3 properties:

1. There is no heat flow, and no viscous forces act. Taking O to be an observer who sees the fluid as possessing no net momentum, this means that

2.2.1 $$T^{\mu\nu} \overset{*}{=} \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \quad ; \quad E^{\mu} \overset{*}{=} (0,0,0,1)$$

Note that in 2.2.1 we have assumed the fluid to be isotropic so that $T^{11} = T^{22} = T^{33} = P$. From 2.2.1 it is clear that

2.2.2 $$T^{\mu\nu} = (\mu + P)E^{\mu} E^{\nu} + Pg^{\mu\nu} .$$

This we will take as our first characteristic of a fluid in equilibrium.

2. Since there is no heat flow, the energy flow vector is parallel to the particle flow vector:

2.2.3 $$M^{\mu} = \rho E^{\mu} .$$

This we will take as our second characteristic of a fluid in equilibrium.

3. There exists a functional relation between the three variable μ, ρ, P :

2.2.4 $$P = f(\mu,\rho) \quad .$$

It follow from the usual postulates of equilibrium thermodynamics that the Pfaffian form

$$d(\frac{\mu}{\rho}) + P(\mu,\rho)\, d(\frac{1}{\rho})$$

has an integrating factor $\frac{1}{T(\mu,\rho)}$ which is strictly positive:

2.2.5 $$T(\mu,\rho)\, d\, S(\mu,\rho) = d(\frac{\mu}{\rho}) + P\, d(\frac{1}{\rho})$$
$$T(\mu,\rho) \geq 0$$

S is defined by 2.2.5. Let us assume that an observer who sees no net particle flow also sees no net entropy flow, i.e. assume

2.2.6 $$S^{\mu} = S\, M^{\mu} \quad .$$

Then equations 2.2.2, 2.2.3 and 2.2.6, plus the conservation equations 2.1.1 and 2.1.2 are sufficient to force the conclusion that S is constant for a fluid in equilibrium:

From 2.2.2 and 2.1.1

2.2.7 $$0 = E_{\mu}\, T^{\mu\nu}_{\ ,\nu} = -\mu_{,\nu}\, E^{\nu} - (\mu + P)\, E^{\nu}_{\ ,\nu} \quad .$$

From 2.2.3 and 2.1.2

2.2.8 $$\rho E^{\nu}{}_{,\nu} = -\rho_{,\mu} E^{\mu} .$$

Substituting 2.2.8 into 2.2.7 gives

2.2.9 $$[\mu_{,\nu} - (\frac{\mu + P}{\rho}) \rho_{,\mu}] E^{\nu} = 0 .$$

Recalling 2.2.6, 2.2.5, 2.2.3 and 2.2.2

2.2.10 $$S^{\mu}{}_{,\mu} = S_{,\nu} \rho E^{\nu}$$
$$= T^{-1} [\mu_{,\nu} - (\frac{\mu + P}{\rho}) \rho_{,\nu}] E^{\nu} .$$

It follows from 2.2.9 that

2.2.11 $$S^{\nu}{}_{,\nu} = 0$$

and hence by 2.2.10

2.2.12 $$S_{,\nu} E^{\nu} = 0 .$$

For a fluid in equilibrium M^{μ} and E^{μ} are parallel, so E^{μ} is the only natural choice for the 4-velocity of the fluid. Thus

2.2.13 $$S_{,\mu} E^{\mu} = \frac{dS}{d\tau} = 0$$

and the change in entropy of the fluid with respect to proper time is zero. Thus we have the important result: If the entropy of a fluid in equilibrium is spatially constant at some initial time, then it is universally constant.

Section 3: Irreversible Thermodynamics.

Here we will consider a fluid far from equilibrium. Since heat flow will, in general, be present, the vectors E^{μ} and M^{μ} will no longer be parallel. There is, then, no natural choice for the fluid 4-velocity. We will retain the postulates of conservation of particles, conservation of energy, and conservation of momentum in the forms

2.3.1 $$T^{\mu\nu}{}_{,\nu} = 0 \quad , \quad M^{\mu}{}_{,\mu} = 0 \ .$$

As dissipative processes are taking place, we shall have in place of $S^{\mu}{}_{,\mu} = 0$,

2.3.2 $$S^{\mu}{}_{,\mu} \geq 0 \ .$$

Let us denote the limit of any quantity Q, as t approaches infinity, by Q_{∞}. If E^{μ} is again defined by

2.3.3 $$T^{\mu\nu} E_{\nu} = -\varphi E^{\mu} \quad ; \quad E^{\mu} E_{\mu} = -1 \ ,$$

it is reasonable to suppose that

2.3.4 $$\lim_{t\to\infty} T^{\mu\nu} = (\mu_{\infty} + P_{\infty}) E^{\mu}_{\infty} E^{\nu}_{\infty} + P_{\infty} g^{\mu\nu}$$

2.3.5 $$\lim_{t\to\infty} M^{\mu} = \rho_{\infty} E^{\mu}_{\infty}$$

2.3.6 $$P_{\infty} = f(\mu_{\infty}, \rho_{\infty}) \ .$$

Let us now choose some network of observers: i.e., define a 4-velocity field U^{μ}

2.3.7 $$U_{\mu} U^{\mu} = -1 \quad ; \quad \lim_{t\to\infty} U^{\mu} = E^{\mu}_{\infty} \ .$$

The second requirement assures us that the network of observers will be those of section 2.2 when equilibrium has been reached.

Now let us define the quantities μ, ζ^{μ}, $S^{\mu\nu}$ by

2.3.8 $$\mu U^{\mu}U^{\nu} + \zeta^{\mu}U^{\nu} + \zeta^{\nu}U^{\mu} + S^{\mu\nu} = T^{\mu\nu}$$

2.3.9 $$\zeta^{\mu}U_{\mu} = 0$$

2.3.10 $$S^{\mu\nu}U_{\nu} = 0$$

The physical interpretation of μ, ζ^{μ} , $S^{\mu\nu}$ is easy to get by considering the equations 8 - 10 in the frame where $U^{\mu} \overset{*}{=} (0,0,0,1)$. It is easy to see that

2.3.11 $$\begin{cases} \zeta^{4} \overset{*}{=} 0 & \zeta^{i} \overset{*}{=} T^{i4} \\ S^{\mu 4} \overset{*}{=} 0 & \\ \mu \overset{*}{=} T^{44} & S^{ij} \overset{*}{=} T^{ij} \end{cases}$$

Hence

$$\mu = \text{energy density}$$
$$\zeta^{\mu} = \text{energy flux (or momentum)}$$
$$S^{\mu\nu} = \text{stress tensor}$$

measured by the chosen network of observers.

Invariant expressions for μ , ζ^{μ} and $S^{\mu\nu}$ may be given using the projection tensor $\Delta^{\mu\nu}$ defined by

2.3.12 $$\Delta^{\mu\nu} = g^{\mu\nu} + U^{\mu} U^{\nu} .$$

It is a simple matter to check that $\Delta^{\mu\nu}$ has the properties

$$\Delta^{\alpha\beta} U_\alpha = 0 \quad , \quad \Delta^\lambda_\alpha \Delta_{\lambda\beta} = \Delta_{\alpha\beta}$$

2.3.13

$$\Delta_{\alpha\beta} \Delta^{\alpha\beta} = 3 \quad , \quad \Delta^{\alpha\beta} A_\alpha A_\beta \geq 0$$

where A_α is any vector. The invariant expressions for μ , ζ^μ , $S^{\mu\nu}$ are then given by

2.3.14
$$\mu = T^{\mu\nu} U_\mu U_\nu$$
$$\zeta^\mu = -T^{\alpha\beta} \Delta^\mu_\alpha U_\beta$$
$$S^{\mu\nu} = T^{\alpha\beta} \Delta^\mu_\alpha \Delta^\nu_\beta$$

The correctness of equations 2.3.14 is easily checked by observing that in the coordinate system where $U^\alpha \overset{*}{=} (0,0,0,1)$ they reduce to the expressions 2.3.11. Since the quantities involved are tensors the equations are correct in all coordinate systems.

Next let us express M^μ in terms of U^α . Define ρ and ξ^μ by

2.3.15 $$M^\mu = \rho U^\mu + \xi^\mu \quad ; \quad \xi^\mu U_\mu = 0 \ .$$

It is clear that

2.3.16 $$\rho = - M^\mu U_\mu$$

and in the coordinate system where $U^\mu \overset{*}{=} (0\ 0\ 0\ 1)$ this becomes

2.3.17 $$\rho \overset{*}{=} M^4 \ .$$

Thus, for the chosen observer, ρ is the number of particles per c.c. multiplied by the rest mass of one particle. In the special coordinate

system $U^{\mu} \overset{*}{=} (0,0,0,1)$

2.3.18 $$\xi^{i} \overset{*}{=} M^{i} \quad ; \quad \xi^{4} \overset{*}{=} 0 \ .$$

Thus ξ^{i} represents the particle flux.

We have now defined μ and ρ, but what about P ? In equilibrium, spatial isotropy and absence of viscosity enable us to define the thermodynamic pressure as equal to any of the three identical measurable stress components $S^{1}_{1} = S^{2}_{2} = S^{3}_{3}$. Away from equilibrium, the concept of thermodynamic pressure becomes nebulous, since we are attempting to apply an idea of equilibrium thermodynamics to a non-equilibrium situation. No longer are S^{1}_{1} , S^{2}_{2} , S^{3}_{3} all equal, since in a non-equilibrium situation bulk viscosity terms appear in these components. It seems that it is impossible to separate reversible (Pressure) effects from irreversible (viscosity) effects except at or near equilibrium. We will therefore adopt a new appraoch: At equilibrium we are supplied with a functional form (The equation of state) for P_{∞} :

2.3.19 $$P_{\infty} = f(\mu_{\infty}, \rho_{\infty}) \ .$$

Let us use this form to define P in the non-equilibrium case:

2.3.20 $$P = f(\mu, \rho)$$

where f is the same functional form as in 2.3.19. Since 2.2.5 only depends on the functional form of P in terms of μ , ρ we will retain the Gibbs relation

2.3.21 $$T(\mu,\rho)\, dS(\mu,\rho) = d\left(\frac{\mu}{\rho}\right) + P d\left(\frac{1}{\rho}\right) \quad ; \quad T \geq 0 \ .$$

Note that μ, ρ, P, T, S, depend on U^{α} . This means that obervers

in different networks U_1^α , U_2^α etc. will observe different values $\mu_1, \rho_1, P_1, T_1, S_1$ and μ_2, ρ_2, P_2, T_2, S_2 at the same event in space-time. Only as $t \to \infty$ will the sets of observations tend to agreement. It is worth noting here that the quantities S , $\frac{\mu}{\rho}$ appearing in 2.3.21 refer to entropy per gram of rest mass and energy per gram of rest mass at a point. If we wish to express the relation in terms of the total entropy S_1 and total energy U_1 of a (small) finite region of volume V_1 , containing rest mass M_1 , we would proceed as follows: By 2.3.21

2.3.22
$$Td\left(\frac{S_1}{M_1}\right) = d\left(\frac{U_1}{M_1}\right) + Pd\left(\frac{V_1}{M_1}\right)$$

or

$$\frac{TdS_1}{M_1} = \frac{TS_1}{M_1^2}\,dM_1 + \frac{1}{M_1}\,dU_1 - \frac{U_1}{M_1^2}\,dM_1 + \frac{P}{M_1}\,dV_1 - \frac{PV_1}{M_1^2}\,dM_1 \quad .$$

Putting

$$\theta = \frac{U_1 + P_1V_1 - TS_1}{T\ M_1} = \text{thermodynamic potential/gm}$$

2.3.23
$$TdS_1 = dU_1 + PdV_1 - T\theta\ dM_1 \quad .$$

This gives the change in entropy of a finite region whose volume may change, and which may experience gain or loss of particles across its boundary. The expression $\frac{\mu + P}{\rho}$ will appear frequently, so for future simplification we will define η , the enthalpy per gram by

2.3.24
$$(\mu + P) = \eta\rho \quad .$$

The Gibbs relation then shows

2.3.25 $$d\eta = TdS + \frac{1}{\rho} dP .$$

Since

2.3.26 $$\theta = \frac{\eta}{T} - S$$

equation 2.3.25 also implies

2.3.27 $$d\theta = \frac{1}{\rho T} dP + \eta d(\frac{1}{T}) .$$

Having defined the thermodynamic pressure P we can now separate the viscous $(\tau^{\mu\nu})$ and non-viscous parts of the stress tensor $S^{\mu\nu}$:

2.3.28 $$S^{\mu\nu} = \tau^{\mu\nu} + P \Delta^{\mu\nu} .$$

Multiplication of 2.3.28 by U^{μ} and use of 2.3.13 and 2.3.10 give

2.3.29 $$\tau^{\mu\nu} U_{\nu} = 0 .$$

We are now prepared to express $T^{\mu\nu}$ in its final form

2.3.30 $$T^{\mu\nu} = \eta\rho U^{\mu} U^{\nu} + Pg^{\mu\nu} + \zeta^{\mu}U^{\nu} + \zeta^{\nu}U^{\mu} + \tau^{\mu\nu} .$$

Let us now investigate the consequences of the conservation equations $T^{\mu\nu}_{\ \ ,\nu} = 0$ and $M^{\mu}_{\ ,\mu} = 0$. Substituting 2.3.30 into $T^{\mu\nu}_{\ \ ,\nu} = 0$ we get

2.3.31 $$(\eta U^{\mu} \rho U^{\nu})_{,\nu} + P_{,\nu} g^{\mu\nu} + (\zeta^{\mu}U^{\nu} + \zeta^{\nu}U^{\mu} + \tau^{\mu\nu})_{,\nu} = 0 .$$

Substituting 2.3.15 into $M^{\mu}_{\ ,\mu} = 0$ we get

2.3.32 $$(\rho U^{\nu})_{,\nu} = \xi^{\nu}_{\ ,\nu} .$$

For brevity, let us write $\dot{\varphi}$ for $\varphi_{,\nu}U^{\nu}$. Substituting 2.3.32 into 2.3.31 gives, with use of 2.3.25

2.3.33
$$(\rho T\dot{S} - \eta\xi^{\nu}_{,\nu} + \zeta^{\nu}_{,\nu})U^{\mu} + \dot{\zeta}^{\mu} + \tau^{\mu\nu}_{,\nu}$$
$$+ \eta\rho\dot{U}^{\mu} + P_{,\nu}g^{\mu\nu} + \dot{P}U^{\mu} + \zeta^{\mu}U^{\nu}_{,\nu} + \zeta^{\nu}U^{\mu}_{,\nu} = 0 .$$

This may be rearranged as

2.3.34
$$TU^{\mu}([SM^{\alpha} + \frac{\zeta^{\alpha} - \eta\xi^{\alpha}}{T}]_{,\alpha} + \theta_{,\alpha}\xi^{\alpha} - (\tfrac{1}{T})_{,\alpha}\zeta^{\alpha})$$
$$+ \dot{\zeta}^{\mu} + \tau^{\mu\nu}_{,\nu} + \eta\rho\dot{U}^{\mu} + P_{,\alpha}\triangle^{\mu\alpha} + \zeta^{\mu}U^{\alpha}_{,\alpha} + U^{\mu}_{,\alpha}\zeta^{\alpha}$$
$$= 0 .$$

Our plan for the immediate future is:

1. To project equation 2.3.34 on the time-like vector U^{α} .

2. To consider the spacelike projection of equation 2.3.34 by multiplying by $\triangle_{\mu\beta}$.

3. To combine the results of 1 and 2 into a useful expression for $S^{\alpha}_{,\alpha}$.

1. First, multiply 2.3.34 by U_{μ} :

2.3.35
$$([SM^{\alpha} + \frac{\zeta^{\alpha} - \eta\xi^{\alpha}}{T}]_{,\alpha} + \theta_{,\alpha}\xi^{\alpha} - (\tfrac{1}{T})_{,\alpha}\zeta^{\alpha})$$
$$- \dot{\zeta}^{\perp}U_{\mu} - \tau^{\mu\nu}_{,\nu}U_{\mu} = 0 .$$

To simplify this observe that equations 2.3.9 and 2.3.29 imply

2.3.36 $$U_\mu \dot{\zeta}^\mu = - \dot{U}^\mu \zeta_\mu$$

2.3.37 $$U_\mu \tau^{\mu\nu}_{,\nu} = - U_{\mu,\alpha} \tau^{\mu\alpha} \quad .$$

Also, to symmetrise the R.H.S. of 2.3.37 observe that if

2.3.38 $$\epsilon_{\alpha\beta} = \tfrac{1}{2}\Delta^\mu_\alpha \Delta^\nu_\beta (U_{\mu,\nu} + U_{\nu,\mu})$$

then

2.3.39 $$U_\mu \tau^{\mu\nu}_{,\nu} = -\epsilon_{\alpha\beta} \tau^{\alpha\beta} \quad .$$

We can then write 2.3.35 as

2.3.40 $$[SM^\alpha + \frac{\zeta^\alpha - \eta\xi^\alpha}{T}]_{,\alpha} = - \frac{1}{T} \epsilon_{\alpha\beta} \tau^{\alpha\beta} - \theta_{,\alpha}\xi^\alpha + [(\frac{1}{T})_{,\alpha}\zeta^\alpha - \frac{1}{T} \dot{U}_\mu \zeta^\mu] \ .$$

2. Now let us consider the spacelike portion of 2.3.34 by multiplying through by Δ^λ_μ . Since

2.3.41 $$\Delta^\lambda_\mu \zeta^\mu = \zeta^\lambda \quad ; \quad \Delta^\lambda_\mu \dot{U}^\mu = \dot{U}^\lambda \quad ; \quad \Delta^\lambda_\mu U^\mu_{,\alpha} = U^\lambda_{,\alpha}$$

we get

2.3.42 $$(\dot{\zeta}^\mu \Delta^\lambda_\mu + \tau^{\mu\nu}_{,\nu}\Delta^\lambda_\mu + \zeta^\lambda U^\alpha_{,\alpha} + U^\lambda_{,\alpha} \zeta^\alpha) + \eta\rho\dot{U}^\lambda + P_{,\alpha} \Delta^{\lambda\alpha} = 0$$

To simplify, put

2.3.43 $$\eta\rho T\delta^\lambda = (\dot{\zeta}^\mu \Delta^\lambda_\mu + \tau^{\mu\nu}_{,\nu} \Delta^\lambda_\mu + \zeta^\lambda U^\alpha_{,\alpha} + U^\lambda_{,\alpha}\zeta^\alpha) \quad .$$

It is obvious that

2.3.44 $$\delta^{\lambda} U_{\lambda} = 0 \quad .$$

We now have the simple expression

2.3.45 $$\eta\rho \dot{U}^{\lambda} = -P_{,\alpha} \Delta^{\lambda\alpha} - \eta\rho T \delta^{\lambda} \quad .$$

To get this in shape to substitute into the timelike projection 2.3.40, multiply it by ζ_{λ} :

2.3.46 $$\dot{U}^{\lambda} \zeta_{\lambda} = - \frac{\zeta^{\alpha} P_{,\alpha}}{\eta\rho} - T\delta^{\alpha} \zeta_{\alpha} \quad .$$

3. We will now combine the spacelike 2.3.46 and timelike projection 2.3.40 of $T^{\mu\nu}_{\ ,\nu} = 0$:

2.3.47 $$\left(SM^{\alpha} + \frac{\zeta^{\alpha} - \eta\xi^{\alpha}}{T}\right)_{,\alpha} = \frac{\theta_{,\alpha}}{\eta} (\zeta^{\alpha} - \eta\xi^{\alpha}) - \frac{1}{T} \epsilon_{\alpha\beta} \tau^{\alpha\beta} + \delta^{\alpha} \zeta_{\alpha} \quad .$$

We must now make an assumption about the relation between the entropy flux vector and the heat flow and particle flow vectors.

If S_1, U_1, M_1 refer to the total entropy, energy, and rest mass inside a finite region of fixed volume we have

2.3.48 $$TdS_1 = dU_1 - T\theta \, dM_1$$

(cf. 2.3.23). We will assume that the fluxes of entropy, energy, and rest mass observed by a man with 4-velocity U^{α} satisfy the same linear relation as do the differentials of entropy, energy and rest mass. i.e.,

2.3.49 $$T(S^{\mu} - S\rho U^{\mu}) = \zeta^{\mu} - T\theta(M^{\mu} - \rho U^{\mu}) \quad .$$

This may be rearranged as

2.3.50 $$S^{\mu} = SM^{\mu} + \frac{\zeta^{\mu} - \eta\xi^{\mu}}{T} \quad .$$

Substituting this into 2.3.47 yields our final equation for the entropy production:

2.3.51 $$S^{\mu}_{,\mu} = \frac{1}{\eta}\,\theta_{,\alpha}\,(\zeta^{\alpha} - \eta\xi^{\alpha}) - \frac{1}{T}\,\epsilon_{\alpha\beta}\tau^{\alpha\beta} + \delta^{\alpha}\,\zeta_{\alpha} \quad .$$

This is an important equation from which we shall derive most of the further results of this chapter. First, observe that when $t \to \infty$, $T^{\mu\nu}$, M^{μ} , S^{μ} are assumed to approach their perfect fluid forms. Thus

2.3.52 $$\lim_{t\to\infty} (\zeta^{\alpha} , \xi^{\alpha} , \tau^{\alpha\beta}) = (0,0,0)$$

Equation 2.3.51 then shows that

2.3.53 $$\lim_{t\to\infty} S^{\alpha}_{,\alpha} = 0 \quad .$$

We were able to show (2.2.9 et seq.) that

2.3.54 $$S^{\alpha}_{\infty;\alpha} = 0$$

so we now have, as expected,

2.3.55 $$\lim_{t\to\infty} S^{\alpha}_{,\alpha} = S^{\alpha}_{\infty,\alpha} \, .$$

Section 4: A Comparison of the Eckart and Landau-Lifshitz Formalisms.

The two formalisms differ in the selection of the network of observers, i.e., in the choice of U^α. Eckart selects U^α parallel to M^α, while Landau and Lifshitz select U^α parallel to E^α. Thus, in Eckart's rest frame, we observe no net particle flow, while in Landau-Lifshitz rest frame, we observe no net energy flow or momentum.

A. The Eckart Formalism.

For Eckart the vector U^α is defined by

2.4.1 $$M^\alpha = \rho U^\alpha \quad .$$

In our previous work this implies

2.4.2 $$\xi^\mu = 0 \quad .$$

The entropy production is then given by 2.3.51

2.4.3 $$S^\alpha_{\ ,\alpha} = \theta_{,\alpha} \frac{1}{\eta} \zeta^\alpha - \frac{1}{T} \epsilon_{\alpha\beta} \tau^{\alpha\beta} - \delta^\alpha \zeta_\alpha \quad .$$

If now we assume:

(i) ζ^α and $\tau^{\alpha\beta}$ are linear functions of $(\frac{1}{\eta} \theta_{,\beta} - \delta_\beta)$ and $\epsilon_{\alpha\beta}$

(ii) that the fluid is isotropic

we are able to show (appendix 1)

2.4.4 $$\tau^{\alpha\beta} = -\lambda_1 \Delta^{\lambda\alpha} \Delta^{\mu\beta}(U_{\lambda,\mu} + U_{\mu,\lambda}) - \lambda_2 \Delta^{\alpha\beta} U^{\lambda}_{,\lambda}$$

$$\lambda_1 \geq 0 \qquad \lambda_2 \geq 0$$

2.4.5 $$\zeta^{\alpha} = K \Delta^{\alpha\beta}(\frac{1}{\eta}\theta_{,\beta} - \delta_{\beta}) \quad ; \quad K \geq 0 \quad .$$

It is worth noting that 2.4.5 can be rewritten by the use of 2.3.45 and 2.3.27 as

2.4.6 $$\zeta^{\alpha} = K \Delta^{\alpha\beta} [(\frac{1}{T})_{,\beta} - \frac{\dot{U}_{\beta}}{T}] \quad K \geq 0 \quad .$$

This is the form of ζ^{α} in Eckart's original paper.

B. Landau-Lifshitz Formalism.

Landau and Lifshitz, for their formalism select

2.4.7 $$U^{\alpha} = E^{\alpha} \quad .$$

The first consequence of this selection is the vanishing of ζ^{μ} :

2.4.8 $$T^{\mu\nu} U_{\nu} = -\varphi U^{\mu} \quad .$$

Multiplication by U_{μ} gives, using 2.3.14, 2.3.7

2.4.9 $$\mu = +\varphi \quad .$$

We then have

$$(\mu U^{\mu}U^{\nu} + P\Delta^{\mu\nu} + \zeta^{\mu}U^{\nu} + \zeta^{\nu}U^{\mu} + \tau^{\mu\nu})U_{\nu} = -\mu U^{\mu}$$

or by virtue of 2.3.9, 2.3.29 and 2.3.7

$$-\mu U^{\mu} - \zeta^{\mu} = -\mu U^{\mu}$$

2.4.10 i.e.

$$\zeta^{\mu} = 0 \ .$$

The Entropy production 2.3.51 then becomes

2.4.11
$$S^{\alpha}_{,\alpha} = -\theta_{,\alpha}\,\xi^{\alpha} - \frac{1}{T}\,\epsilon_{\alpha\beta}\,\tau^{\alpha\beta}$$

If as before we assume:

(i) ζ^{α} and $\tau^{\alpha\beta}$ are linear functions of $\theta_{,\beta}$ and $\epsilon_{\alpha\beta}$

(ii) the fluid is isotropic

we are able to show (Appendix 1)

2.4.12
$$\tau^{\alpha\beta} = -\lambda_1\,\Delta^{\lambda\alpha}\,\Delta^{\mu\beta}(U_{\lambda,\mu} + U_{\mu,\lambda}) - \lambda_2\,\Delta^{\alpha\beta}\,U^{\lambda}_{,\lambda} \quad ; \quad \lambda_1 \geq 0 \quad ; \quad \lambda_2 \geq 0$$

2.4.13
$$\xi^{\alpha} = -K\,\Delta^{\alpha\beta}\theta_{,\beta} \qquad K \geq 0 \ .$$

The similarities between the Eckart results and the Landau-Lifshitz results are striking. The only difference appears in equations 2.4.11 and 2.4.3 ; the term $\delta_{\alpha}\,\zeta^{\alpha}$ appears in the former and not in the latter. It is worthwhile investigating the magnitude of this term. Let us use a 3D formalism, and drop the convention $c = 1$. To be specific, let us:

a. use a coordinate system wherein $U^{\alpha} \stackrel{*}{=} (0,0,0,c)$

b. $\vec{V} = (U^1, U^2, U^3)$

c. measure μ in gm/c.c.

d. measure P in dynes/sq. cm.

e. measure ρ in gms/c.c.

f. measure T in ^{o}K

g. measure ζ^i in $\frac{\text{dyn-cm.}}{\text{sq. cm.}}$ per sc $= \frac{\text{dyne}}{\text{cm sec.}}$

h. measure ζ^{λ} in $\frac{\text{cm}}{\text{sec}^2\ {}^{o}K}$

i. measure K in $^{o}K\ \frac{\text{dyne}}{\text{sec}}$.

Immediate results of these conventions are:

2.4.14 $$\eta = \frac{\mu + P/c^2}{\rho}$$

is dimensionless;

2.4.15 $$\theta = \frac{\eta}{T} - \frac{S}{c^2}$$

has dimensions $(^{o}K)^{-1}$.

A straightforward dimensional analysis of equation 2.3.43 then yields

2.4.16 $$c^2 \eta\rho T\zeta^{\lambda} = \vec{\zeta}\,\mathrm{div}\,\vec{V} + (\vec{\zeta}\cdot\vec{\nabla})\vec{V} + c^2\,\mathrm{div}\,\vec{\tau} + (\vec{V}\cdot\vec{\nabla})\,\vec{\zeta}$$

A similar analysis applied to 2.4.8 gives

2.4.17 $$\vec{\zeta} = K(\mathrm{grad}\,\theta + \frac{1}{c^2}\vec{\delta}) \quad .$$

Under what conditions are we assured that grad θ dominates $\frac{1}{c^2}\vec{\delta}$? This will be the case if

2.4.18 $$\frac{1}{c^2}\vec{\delta} << \mathrm{grad}\,\theta$$

or

2.4.19 $$\vec{\zeta} \approx K\,\mathrm{grad}\,\theta \quad .$$

We know (cf. Taub Phys. Rev. 74 , 328 (1948)) that $P \leq \frac{1}{3}\mu c^2$ for all "reasonable" fluids. Thus

2.4.20 $$\mu \leq \eta\rho \leq \frac{4}{3}\mu$$

$$\eta\rho \sim \mu \quad .$$

Consideration of 2.4.24, 2.4.27, 2.4.28 shows that $\frac{1}{c^2}\vec{\delta}$ is negligible provided

$$\left| \frac{K \text{ grad } \theta \text{ div } \vec{V}}{c^4 \mu T} \right| << | \text{grad } \theta |$$

$$\left| \frac{K(\text{grad } \theta \cdot \nabla)\vec{V}}{c^4 \mu T} \right| << | \text{grad } \theta |$$

2.4.22

$$\left| \frac{K(\vec{V} \cdot \nabla) \text{ grad } \theta}{c^4 \mu T} \right| << | \text{grad } \theta |$$

$$\left| \frac{\text{div } \vec{\vec{\tau}}}{c^2 \mu T} \right| << | \text{grad } \theta |$$

Under classical conditions, the high powers of c in 2.4.22 guarantee satisfaction. There is reason to believe that the validity of the formalism is restricted to cases where 2.4.22 applies. In these cases $\delta^\alpha \zeta_\alpha$ is negligible and the expressions for entropy production are formally identical in Eckart and Landau-Lifshitz. The same results could be ensured by restricting ourselves to fluids for which K, λ_1, λ_2 are sufficiently small. If we then retain in our expression for $S^\alpha_{,\alpha}$ only those terms which are linear in K, λ_1, λ_2 the term $S_\alpha \zeta^\alpha$ will be dropped. (It is clear from 2.3.43, 2.4.8, 2.4.7 that $\delta^\alpha \zeta_\alpha$ is quadratic in K, λ_1, λ_2 .) It is obviously a matter of choice whether we regard the inequalities 2.4.22 as restricting the size of the coefficients or as restricting the permissible gradients.

To summarize our comparison of Landau-Lifshitz and Eckart (as herein elaborated and interpreted):

A. Because of different rest frames μ(ECKART), ρ(ECKART)etc.

are different from μ(L-L), ρ(L-L) etc.

B. Both use the same functional form for P .

C. The final expressions for the entropy production are formally identical, to within the approximation 2.4.22.

D. The question of equivalence is still not settled, due to the (unknown as yet) differences in μ(ECKART),μ(L-L), etc. This question will be answered in the next section, where we will introduce:

Section 5: A New, General, Formalism for Irreversible Thermodynamics.

We will leave the network of observers arbitrary, but will consider only the case where t is moderately large - the case where the fluid is reasonably close to equilibrium. It is necessary, of course, to specify what we mean by "moderately large" and "reasonably close". This presents our first difficulty.

One may try to specify its closeness by requiring that the viscous forces should be small compared to the pressure, e.g.

2.5.1 $$\frac{\sqrt{\tau_{ij}\ \tau^{ij}}}{P} << 1 .$$

One also needs a restriction on the heat flow; for example it may be required that the thermal energy flux be small compared to the mass energy flux:

$$\vec{q} << \mu c^2 \cdot \vec{v} < \mu c^3 \qquad \text{or}$$

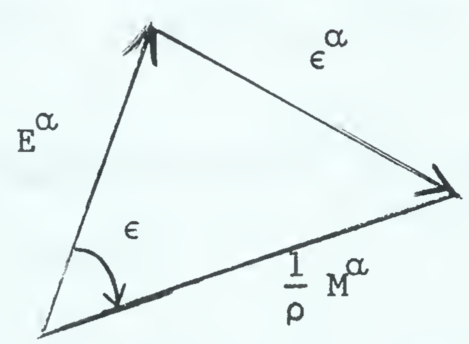
2.5.2
$$\frac{\vec{q}}{\mu c^3} << 1 \quad .$$

This requirement is met automatically in the classical limit where $c \to \infty$, but it places a definite restriction on the range of validity of our relativistic theory.

We however, will give these restrictions more geometric form. Near equilibrium, the vectors E^α and M^α are near their equilibrium positions, i.e., nearly parallel. Let

2.5.3
$$E^\alpha - \frac{1}{\rho} M^\alpha = \epsilon^\alpha(x_1\ x_2\ x_3\ x_4) \quad ; \quad \epsilon = |\epsilon^\alpha|$$

ϵ is dimensionless and we will take it as our standard of smallness. It can be thought of as the pseudo-angle between E^α and M^α



We shall think of ϵ as negligible in comparison with $\frac{P}{\mu c^2}$. Since (cf. Taub) $P < \frac{1}{3} \mu c^2$ for all "reasonable" fluids we have

2.5.4
$$\epsilon << \frac{P}{\mu c^2} < 1 \ .$$

The cone obtained by rotating E^α and M^α about their bisector will be referred to as the ϵ cone at the event P under consideration.

Since we are going to consider the fluid only near equilibrium, we will require that the gradient of ϵ^α is small in the sense that $\epsilon^\alpha{}_{,\beta} \big/ \epsilon << 1$. More precisely, in analogy with 2.4.29 we will assume that

2.5.5 $$\frac{K}{c^4 \mu T} \frac{|\dot{\vec{\epsilon}}|}{\epsilon} << 1 \quad , \quad \frac{K}{c^4 \mu T} \frac{|\nabla \vec{\epsilon}|}{\epsilon} << 1 \ .$$

Here K is Fouriers coefficient. The restrictions 2.5.2 insure that the ϵ cone does not change abruptly in shape or orientation from point to point. Let us summarise our restrictions:

A. We allow an arbitrary network of observers, characterised by a vector field $U^\alpha(x_1\ x_2\ x_3\ x_4)$, subject to

2.5.6 $$|U^\alpha - \frac{1}{\rho} M^\alpha| = |\frac{1}{\rho} \xi^\alpha| = O(\epsilon)$$

2.5.7 $$|U^\alpha - E^\alpha| = |\frac{1}{\mu} \zeta^\alpha| = O(\epsilon) \ .$$

B. The orientation of U^α relative to the ϵ cone should not vary abruptly from point to point. To wit:

2.5.8 $$\frac{K}{\mu T} \frac{|\xi^\alpha{}_{,\beta}|}{|\xi^\alpha|} << 1 \quad ; \quad \frac{K}{\mu T} \frac{|\zeta^\alpha{}_{,\beta}|}{|\zeta^\alpha|} << 1 \ .$$

C. The viscosity tensor should not charge abruptly from point to point

2.5.9 $$\frac{|\tau^{\alpha\beta}{}_{,\gamma}|}{\mu T \epsilon} << 1 \ .$$

We have then, an infinite set of possible networks of observers, including Eckart and Landau-Lifshitz, each characterised by a vector field

U^{α} subject to 2.5.3, 2.5.4, 2.5.6, 2.5.7. Any observer can decompose $T^{\mu\nu}$ according to 2.3.30.

As a first consequence of these restrictions, let us show that if the viscous stress tensor is small in the eyes of one observer, it is small for all observers. To be specific, assume that for one observer (U^{α})

2.5.10 $$\tau^{\mu\nu} = O(\epsilon_1 P) \quad ;$$

Consider another observer with velocity U_*^{α} given by

2.5.11 $$U^{\alpha} = U_*^{\alpha} + \lambda^{\alpha} \quad .$$

The requirements 2.5.6, 2.5.7 imply

2.5.12 $$\lambda^{\alpha} = O(\epsilon) \quad .$$

Now 2.5.12 implies that

2.5.13 $$(U^{\alpha} - U_*^{\alpha})(U_{\alpha} - U_{*\alpha}) = O(\epsilon^2) \quad .$$

Thus

2.5.14 $$U_*^{\alpha} U_{\alpha} = -1 + O(\epsilon^2) \quad .$$

We then have

2.5.15 $$U_*^{\alpha} \lambda_{\alpha} = O(\epsilon^2)$$

2.5.16 $$U^{\alpha} \lambda_{\alpha} = O(\epsilon^2) \quad .$$

The observer with velocity U^α may decompose M^α into $\rho U^\alpha + \xi^\alpha$. We then have

2.5.17 $$M^\alpha U_{*\alpha} = (\rho U_*{}^\alpha + \rho\lambda^\alpha + \xi^\alpha)\, U_{*\alpha} \quad .$$

Since $\rho_* = -M^\alpha U_{*\alpha}$ we have established

2.5.18 $$\rho_* = \rho + O(\epsilon^2\rho) \quad .$$

Here we have used 2.5.6, 2.5 7, 2.5.12 and 2.3.16.

The observer with velocity $U_*{}^\alpha$ may also decompose M^α into $\rho_* U_*^\alpha + \xi_*^\alpha$. In view of 2.5.15 we may write

2.5.19 $$M^\alpha = \rho U_*{}^\alpha + \xi_*{}^\alpha + O(\rho\epsilon^2) \quad .$$

A comparison of 2.5.16 and

2.5.20 $$M^\alpha = \rho U^\alpha + \xi^\alpha$$

yields

$$\xi_*{}^\mu = \xi^\mu + \rho\lambda^\mu + O(\rho\epsilon^2)$$

Consider now the decomposition $T^{\mu\nu} = \mu U^\mu U^\nu + P\Delta^{\mu\nu} + \zeta^\mu U^\nu + \zeta^\nu U^\mu + \tau^{\mu\nu}$ Substitution of 2.5.11 into this expression yields

2.5.21 $$\begin{aligned} T^{\mu\nu} &= \mu U_*{}^\mu U_*{}^\nu + P\Delta_*^{\mu\nu} + \tau^{\mu\nu} + O(\epsilon^2\mu) \\ &\quad + [\zeta^\mu + (\mu+P)\lambda^\mu]U_*{}^\nu + [\zeta^\nu + (\mu+P)\lambda^\nu]U_*{}^\mu \quad . \end{aligned}$$

If we multiply 2.5.21 by $U_{*\mu}U_{*\nu}$ we get (by 2.3.14)

2.5.22 $$\mu_* = \mu + O(\epsilon^2\mu) \quad .$$

If we multiply 2.5.21 by $\Delta_{*\mu}^{\alpha} U_{*\nu}$ we get (by 2.3.14)

2.5.23 $$\zeta_*^{\alpha} = \zeta^{\alpha} + (\mu + P)\lambda^{\alpha} + O(\epsilon^2\mu) \quad .$$

If we take the equation of state

2.5.24 $$P = P(\mu,\rho) \quad ,$$

we may in view of 2.5.22 and 2.5.18 write

2.5.25 $$P_* = P + O(\epsilon^2\mu) + O(\epsilon^2\rho) \quad .$$

It may be shown (Appendix 2) that

2.5.26 $$O(\epsilon^2\mu) + O(\epsilon^2\rho) = O(\epsilon^2 P) \quad .$$

Hence we have

2.5.27 $$P_* = P + O(\epsilon^2 P) \quad .$$

If we multiply 2.5.15 by $\Delta_{*\mu}^{\alpha} \; \Delta_{*\nu}^{\beta}$ we get

2.5.28 $$\tau_*^{\alpha\beta} = \tau^{\alpha\beta} + O(\epsilon^2 P) \quad .$$

We have thus proved that if the viscosity tensor is small for one observer, it is small for any observer. We have also proved a number of other significant results regarding the differences in μ , ρ, P etc. as observed by different observers.

Let us collect them together:

2.5.29
$$\mu_* = \mu + O(\epsilon^2\mu) \quad , \quad \rho_* = \rho + O(\epsilon^2\rho)$$
$$P_* = P + O(\epsilon^2 P) \quad , \quad \tau_*^{\mu\nu} = \tau^{\mu\nu} + O(\epsilon^2 P)$$
$$\zeta_*^\mu = \zeta^\mu + (\mu+P)\lambda^\mu + O(\epsilon^2\mu)$$
$$\xi_*^\mu = \xi^\mu + \rho\,\lambda^\mu + O(\epsilon^2\rho)$$

It is significant that neither ζ^μ nor ξ^μ is invariant under changes in the network of observers. We can, however, make up an invariant quantity from the two. If we define

2.5.30
$$q^\mu = \xi^\mu - \frac{1}{\eta}\zeta^\mu$$

it is clear from the last two equations in 2.5.29 that

2.5.31
$$q_*^\mu = q^\mu + O(\epsilon^2\rho) \quad .$$

If we consider q^μ in the Landau-Lifshitz frame $(U^\mu = E^\mu)$ we have $\zeta^\mu = 0$, or

2.5.32
$$q^\mu \overset{*}{=} \xi^\mu \quad .$$

Thus q^μ represents the particle flux relative to the energy flux. Since q^μ is independent of the choice of U^μ it must always represent the particle flux relative to the energy flux. Further, in this frame $U^\alpha = E^\alpha$, so by 2.5.3 and 2.3.15

2.5.33 $$q^{\alpha} = -\rho\epsilon^{\alpha} .$$

It is worth noting that 2.5.32 shows that our assumption 2.3.49 is the same assumption for all observers: In the equivalent form 2.3.50 all quantities are independent of U^{μ} . We may now write the expression for $S^{\alpha}_{,\alpha}$ in its final form:

2.5.34 $$S^{\alpha}_{,\alpha} = -\theta_{,\alpha} q^{\alpha} - \frac{1}{T} \epsilon_{\alpha\beta} \tau^{\alpha\beta} + \delta^{\alpha} \zeta_{\alpha} .$$

If we admit the inequalities 2.4.22 (or if we think of K , λ_1 , λ_2 as "small" and include only those terms which would be linear in them), $\delta^{\alpha}\zeta_{\alpha}$ is negligible and we have

2.5.35 $$S^{\alpha}_{,\alpha} = -\theta_{,\alpha} q^{\alpha} - \frac{1}{T} \epsilon_{\alpha\beta} \tau^{\alpha\beta} .$$

The requirement that $S^{\alpha}_{,\alpha}$ be non-negative, plus the assumptions of linearity and isotropy then yield (See Appendix 1)

2.5.36 $$\tau^{\mu\nu} = -\lambda_1 \Delta^{\alpha\mu} \Delta^{\beta\nu} \epsilon_{\alpha\beta} - \lambda_2 \Delta^{\mu\nu} \Delta^{\alpha\beta} \epsilon_{\alpha\beta}$$

$$; \quad \lambda_1 \geq 0 \quad ; \quad \lambda_2 \geq 0$$

2.5.37 $$q^{\alpha} = -K \theta_{,\beta} \Delta^{\alpha\beta} \quad ; \quad K \geq 0 .$$

The formula 2.5.37 is worth commenting upon. In the classical case we have <u>two</u> laws: Fick's Law for particle diffusion

2.5.38 $$\vec{\xi} = - \frac{K}{\eta} \text{ grad } \theta$$

and Fouriers law for heat conduction

2.5.39 $$\vec{\zeta} = - \frac{K}{T^2} \text{ grad } T \quad .$$

In relativity we have a <u>single</u> law $q^{\alpha} = - K \, \theta_{,\beta} \, \Delta^{\alpha\beta}$, of which 2.5.30 and 2.5.31 are but different aspects (observed by different people). The relativistic law thus unifies the two previously unrelated results.

CHAPTER III

EXTENSION OF THE FORMALISM TO A MIXTURE OF N CHARGED FLUIDS IN AN ELECTROMAGNETIC FIELD.

Section 1: Introduction.

We shall assume our mixture to be described by N particle flow vectors M^μ_A , $A = 1, 2, \ldots N$, an energy momentum tensor $T^{\mu\nu}$ and a skew-symmetric electromagnetic field tensor $F^{\mu\nu}$. We will assume that the ratio of charge to rest mass of each particle of fluid A is e_A . If we select a network of observers by defining a vector field U^α we may write, as in Chapter II

3.1.1 $$M^\mu_A = \rho_A U^\mu + \xi^\mu_A \quad ; \quad \xi^\mu_A U_\mu = 0 \ .$$

Then

3.1.2 $$- M^\mu_A U_\mu = \rho_A \ .$$

Let us define the total particle flux by

3.1.3 $$M^\mu = \sum_A M^\mu_A \ .$$

If we define ρ and ξ^μ by

3.1.4 $$M^\mu = \rho U^\mu + \xi^\mu \quad ; \quad \xi^\mu U_\mu = 0$$

it is not hard to show that

3.1.5 $$-M^{\mu}U_{\mu} = \rho \quad ; \quad \rho = \sum_A \rho_A \quad ; \quad \xi^{\mu} = \sum_A \xi^{\mu}_A \ .$$

The current flow associated with fluid A is

3.1.6 $$J^{\alpha}_A = e_A M^{\alpha}_A \ .$$

The total current flow is then

3.1.7 $$J^{\alpha} = \sum_A J^{\alpha}_A \ .$$

Let us define the electromagnetic energy momentum tensor $T^{\alpha\beta}_{(E)}$ by

3.1.8 $$T^{\alpha\beta}_{(E)} = g_{\mu\nu}F^{\alpha\mu}F^{\beta\nu} - \frac{1}{4} g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \ .$$

In view of Maxwell's equations(cf. Panofsky and Phillips : Classical Electricity and Magnetism, Addison Wesley)

3.1.9 $$F^{\alpha\beta}{}_{,\beta} = J^{\alpha}$$

and

3.1.10 $$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$$

we may write

3.1.11 $$T_{(E),\mu}{}^{\lambda\mu} = -F^{\lambda\mu}J_{\mu} \ .$$

If we now define

3.1.12 $$T^{\mu\nu}_{(M)} = T^{\mu\nu} - T^{\mu\nu}_{(E)} .$$

We may define μ , ζ^{μ} , $S^{\mu\nu}$ by

$$\mu U^{\mu} U^{\nu} + \zeta^{\mu} U^{\nu} + \zeta^{\nu} U^{\mu} + S^{\mu\nu} = T^{\mu\nu}_{(M)}$$

3.1.13 $$\zeta^{\mu} U_{\mu} = 0$$

$$S^{\mu\nu} U_{\nu} = 0$$

As in Chapter II, invariant expressions for μ , ζ^{μ} , $S^{\mu\nu}$ are

$$\mu = T^{\mu\nu}_{(M)} U_{\mu} U_{\nu}$$

3.1.14 $$\zeta^{\mu} = -T^{\alpha\beta}_{(M)} \Delta^{\mu}_{\alpha} U_{\beta}$$

$$S^{\mu\nu} = T^{\alpha\beta}_{(M)} \Delta^{\mu}_{\alpha} \Delta^{\nu}_{\beta}$$

At equilibrium we have an equation of state

3.1.15 $$P_{\infty} = f(\mu_{\infty}, \rho_{1_{\infty}}, \rho_{2_{\infty}}, \ldots, \rho_{n_{\infty}}) .$$

As in Chapter II we will use the functional form f to define the thermodynamic pressure in the non equilibrium case:

3.1.16 $$P = f(\mu, \rho_1, \rho_2, \ldots \rho_n) .$$

Having defined P we may define the viscous stress tensor $\tau^{\mu\nu}$ by

3.1.17 $$\tau^{\mu\nu} = S^{\mu\nu} - P \Delta^{\mu\nu} .$$

We may now write $T^{\mu\nu}_{(M)}$ in the final form:

3.1.18 $$T^{\mu\nu}_{(M)} = (\mu + P)U^{\mu}U^{\nu} + Pg^{\mu\nu} + \zeta^{\mu}U^{\nu} + \zeta^{\nu}U^{\mu} + \tau^{\mu\nu} .$$

Section 2: Thermodynamic Relations.

At equilibrium we have the Gibbs relation

3.2.1 $$TdS = d\left(\frac{\mu}{\rho}\right) + Pd\left(\frac{1}{\rho}\right) - \sum_A T\theta_A \, d\left(\frac{\rho_A}{\rho}\right) .$$

Since its form is dependent only on f (cf. 3.1.15) we will retain it even under non-equilibrium conditions. If we define η and θ by

3.2.2 $$\eta = \frac{\mu + P}{\rho} \quad ; \quad \theta = \frac{\eta}{T} - S \quad ,$$

we may show that

3.2.3 $$d\eta = TdS + \frac{1}{\rho}\, dP + \sum_A T\theta_A d\left(\frac{\rho_A}{\rho}\right)$$

and

3.2.4 $$d\theta = \eta d\left(\frac{1}{T}\right) + \frac{1}{\rho T}\, dP + \sum_A \theta_A d\left(\frac{\rho_A}{\rho}\right) .$$

It may be shown that

3.2.5 $$\rho\theta = \sum_A \rho_A\theta_A .$$

Since the proof is rather tedious we have placed it in appendix 3.

Section 3: Conservation equations.

We will take as postulates

3.3.1 $$T^{\mu\nu}_{\ \ ,\nu} = 0$$

3.3.2 $$S^{\mu}_{\ ,\mu} \geq 0 \ .$$

Note that we do <u>not</u> require $M^{\mu}_{\ ,\mu} = 0$: We will explicitly include the possibility of nuclear reactions leading to a loss or gain of rest mass.

In view of equations 3.1.11, 3.1.12 and 3.3.1 we may write

3.3.3 $$T_{(M),\mu}^{\lambda\mu} = F^{\lambda\mu} J_{\mu} \quad .$$

To connect the entropy flux vector with the heat flow and particle flow vectors we will assume as in Chapter II that the vectors satisfy the same linear relation as the differentials (3.2.1):

$$\begin{aligned} T(\text{Observed entropy flux}) &= (\text{Observed heat flux}) \\ &\quad - \sum_A T\theta_A \begin{pmatrix}\text{Observed flux} \\ \text{of fluid A}\end{pmatrix} \end{aligned}$$

i.e.,

3.3.4 $$T(S^{\mu} - S\rho U^{\mu}) = \zeta^{\mu} - \sum_A T\theta_A (M_A^{\mu} - \rho_A U^{\mu}) \quad .$$

We will see (Section 4) that this assumption does not depend on the choice of U^μ . This may be written (cf. 3.1.1)

3.3.5 $$T(S^\mu - S\rho U^\mu) = \zeta^\mu - \sum_A T\theta_A \xi^\mu_A$$

or

3.3.6 $$S^\mu = S\rho U^\mu + \frac{1}{T}\zeta^\mu - \sum_A \theta_A \xi^\mu_A \quad .$$

Let us now consider the consequences of the conservation equations $T^{\mu\nu}_{\ ,\nu} = 0$. Using 3.3.3, 3.1.18, 3.2.1 we may write

3.3.7 $$(\rho TS_{,\nu}U^\nu + \eta M^\nu_{\ ,\nu} - \eta\xi^\nu_{\ ,\nu} + \zeta^\nu_{\ ,\nu})U^\mu + \dot{\zeta}^\mu + \tau^{\mu\nu}_{\ ,\nu} + \eta\rho\dot{U}^\mu + P_{,\nu}\Delta^{\mu\nu}$$
$$+ \rho T \sum_A \theta_A \left(\frac{\rho_A}{\rho}\right)_{,\nu} U^\nu U^\mu + \zeta^\mu U^\nu_{\ ,\nu} + \zeta^\nu U^\mu_{\ ,\nu}$$
$$= F^{\mu\lambda} J_\lambda \quad .$$

With the help of equations 3.2.5, 3.1.4, 3.1.1 and a bit of algebra we may write

3.3.8 $$\rho \sum_A \theta_A \left(\frac{\rho_A}{\rho}\right)_{,\nu} U^\nu = -\theta M^\nu_{\ ,\nu} + \theta\xi^\nu_{\ ,\nu} + \sum_A \theta_A M^\nu_{A,\nu} - \sum_A \theta_A \xi^\nu_{A,\nu} \quad .$$

Substitution of 3.3.8 into 3.3.7 yields

3.3.9 $$TU^{\mu}\left\{\rho S_{,\nu}U^{\nu} + (\tfrac{1}{T}\zeta^{\nu})_{,\nu} + S(M^{\mu}_{,\mu} - \xi^{\mu}_{,\mu}) - (\sum_{A} \theta_A \xi^{\nu})_{,\nu}\right\}$$

$$- T(\tfrac{1}{T})_{,\nu}\zeta^{\nu}U^{\mu} + \tau^{\mu\nu}_{,\nu} + \eta\rho\dot{U}^{\mu} + P_{,\nu}\Delta^{\mu\nu} + \zeta^{\mu}U^{\nu}_{,\nu} + \dot{\zeta}^{\mu}$$

$$+ T\sum_{A} \theta_{A,\nu}\xi^{\nu}U^{\mu} + \zeta^{\nu}U^{\mu}_{,\nu} = F^{\mu\lambda}J_{\lambda} - \sum_{A} T\theta_A M^{\nu}_{A,\nu}U^{\mu} .$$

If we take the divergence of 3.3.6 we get

3.3.10 $$S^{\mu}_{,\mu} = \rho S_{,\nu}U^{\nu} + (\tfrac{1}{T}\zeta^{\nu})_{,\nu} + S(M^{\mu}-\xi^{\mu})_{,\mu} - (\sum_{A} \theta_A \xi^{\nu})_{,\nu} .$$

In view of 3.3.10, equation 3.3.9 may be written

3.3.11 $$TU^{\mu}(S^{\nu}_{,\nu}) - T(\tfrac{1}{T})_{,\nu}\zeta^{\nu}U^{\mu} + \tau^{\mu\nu}_{,\nu} + \eta\rho\dot{U}^{\mu} + P_{,\nu}\Delta^{\mu\nu} + \dot{\zeta}^{\mu}$$

$$+ \zeta^{\mu}U^{\nu}_{,\nu} + T\sum_{A} \theta_{A,\nu}\xi^{\nu}_{A}U^{\mu} + \zeta^{\nu}U^{\mu}_{,\nu}$$

$$= F^{\mu\lambda}J_{\lambda} - \sum_{A} T\theta_A M^{\nu}_{A,\nu}U^{\mu} .$$

We will next:

(a) Project 3.3.11 on U_{μ}

(b) Project 3.3.11 on $\Delta_{\mu\alpha}$

(c) Combine the results of (a) and (b) to get a simple expression for $S^{\alpha}_{,\alpha}$.

(a) Multiply 3.3.11 by U_μ . The result is

3.3.12
$$-TS^{\nu}_{,\nu} + T(\tfrac{1}{T})_{,\nu}\zeta^{\nu} - T \sum_{n} \theta_{A,\nu}\xi^{\nu}_{A}$$

$$+\dot{\zeta}^{\mu}U_{\mu} + \tau^{\mu\nu}_{,\nu}U_{\mu} = F^{\mu\lambda}J_{\lambda}U_{\mu} + \sum_{A} T\theta_{A} M^{\nu}_{A,\nu}$$

or in view of

3.3.13
$$U_{\mu}\dot{\zeta}^{\mu} = -\dot{U}^{\mu}\zeta_{\mu} \quad ; \quad U_{\mu}\tau^{\mu\nu}_{,\nu} = -\epsilon_{\alpha\beta}\tau^{\alpha\beta}$$

cf.(2.3.36, 2.3.37) we may put

3.3.14
$$S^{\mu}_{,\mu} = (\tfrac{1}{T})_{,\nu}\zeta^{\nu} - \sum_{A} \theta_{A,\nu}\xi^{\nu}_{A} - \frac{1}{T}\dot{U}^{\mu}\zeta_{\mu} - \frac{1}{T}\tau^{\lambda\mu}\epsilon_{\lambda\mu}$$

$$-\frac{1}{T}F^{\mu\lambda}J_{\lambda}U_{\mu} - \sum \theta_{A} M^{\nu}_{A,\nu} \quad .$$

(b) Multiplying 3.3.11 by Δ^{λ}_{μ} we get

3.3.15
$$(\tau^{\mu\nu}_{,\nu}\Delta^{\lambda}_{\mu} + \dot{\zeta}^{\mu}\Delta^{\lambda}_{\mu} + \zeta^{\lambda}U^{\nu}_{,\nu} + U^{\lambda}_{,\nu}\zeta^{\nu}) + \eta\rho\dot{U}^{\lambda} + P_{,\nu}\Delta^{\lambda\nu}$$

$$= F^{\mu\nu}J_{\nu}\Delta^{\lambda}_{\mu} \quad .$$

Putting

3.3.16
$$\eta\rho T\,\delta^{\lambda} = \dot{\zeta}^{\mu}\Delta^{\lambda}_{\mu} + \zeta^{\lambda}U^{\nu}_{,\nu} + U^{\lambda}_{,\nu}\zeta^{\nu} + \tau^{\mu\nu}_{,\nu}\Delta^{\lambda}_{\mu}$$

we have, upon multiplication of 3.3.15 by ζ_λ

3.3.17 $$\dot{U}^{\lambda}\zeta_{\lambda} = \frac{\zeta^{\nu}P_{,\nu}}{\eta\rho} - T\delta^{\nu}\zeta_{\nu} + \frac{F^{\mu\nu}J_{\nu}\Delta^{\lambda}_{\mu}\zeta_{\lambda}}{\eta\rho} .$$

(c) We will now combine 3.3.14 and 3.3.17 to get an expression for $S^{\mu}_{,\mu}$:

3.3.18 $$S^{\mu}_{,\mu} = (\tfrac{1}{T})_{,\nu}\zeta^{\nu} - \sum_{A} \theta_{A,\nu}\xi^{\nu}_{A} + \frac{P_{,\nu}}{\eta\rho T}\zeta^{\nu} + \delta^{\nu}\zeta_{\nu} - \frac{F^{\mu\nu}J_{\nu}\Delta^{\lambda}_{\mu}\zeta_{\lambda}}{\eta\rho T}$$

$$- \frac{F^{\mu\nu}J_{\nu}U_{\mu}}{T} - \frac{1}{T}\tau^{\lambda\mu}\epsilon_{\lambda\mu} - \sum \theta_{A} M^{\nu}_{A,\nu} .$$

Section 4: Invariance under changes of U^{α}.

As in Chapter I, let us define ϵ by

3.4.1 $$|E^{\alpha} - \frac{1}{\rho} M^{\alpha}| = \epsilon .$$

We will assume that

3.4.2 $$|U^{\alpha} - \frac{1}{\rho_A} M^{\alpha}_{A}| = |\frac{1}{\rho_A}\xi^{\mu}_{A}| = O(\epsilon)$$

3.4.3 $$|U^{\alpha} - E^{\alpha}| = |\frac{1}{\mu}\zeta^{\mu}| = O(\epsilon)$$

To this we will for consistency add the requirement

3.4.4 $$F^{\mu\nu} = O(\epsilon) \; ; \quad \tau^{\mu\nu} = O(\epsilon) .$$

If this were not the case, electrons and protons would be accelerated powerfully in opposite directions, $M^{\alpha}_{Electrons}$ and $M^{\alpha}_{Protons}$ would differ by an appreciable amount, and 3.4.2 could not be satisfied. This requirement shows that to within $O(\epsilon^2)$

3.4.5 $$F^{\mu\nu} J_\mu U_\nu$$

is independent of rest frame.

It is easy to see that under a change of rest frame

3.4.6 $$U^\alpha = U^\alpha_* + \lambda^\alpha ,$$

the quantities T, S, ρ_A, θ_A, μ are invariant within $O(\epsilon^2)$ while ζ^μ, ξ^μ_A transform as

3.4.7 $$\begin{aligned} \zeta^\mu_* &= \zeta^\mu + (\mu+P)\lambda^\mu + O(\epsilon^2\mu) \\ \xi^\mu_{*A} &= \xi^\mu_A + \rho_A \lambda^\mu + O(\epsilon^2\rho) \end{aligned}$$

Thus, dropping terms in $O(\epsilon^2)$

$$\Big(S\rho U^\mu + \frac{1}{T}\zeta^\mu - \sum_A \theta_A \xi^\mu_A\Big)_* = S\rho U^\mu - S\rho\lambda^\mu + \frac{1}{T}\zeta^\nu + \frac{\mu+P}{T}\lambda^\mu$$

$$- \sum_A \theta_A \xi^\mu_A - \sum \theta_A \rho_A \lambda^\mu$$

or in view of 3.2.2, 3.2.5

3.4.8 $$(S\rho U^{\mu} + \tfrac{1}{T}\zeta^{\mu} - \sum_{A} \theta_{A}\xi^{\mu}_{A})_{*} = S\rho U^{\mu} + \tfrac{1}{T}\zeta^{\mu} - \sum_{A} \theta_{A}\xi^{\mu}_{A} \quad .$$

Our assumption about the form of the entropy flux vector is thus undependent of the choice of U^{α} . Every quantity which went into the equation 3.3.18 was independent of U^{α} , so the R.H.S. of 3.3.18 must be independent of U^{α} . It is illuminating to transform 3.3.18 into a form which is obviously independent of U^{α} .

First, observe that in view of 3.4.2, the quantity

3.4.9 $$q^{\mu}_{A} = \xi^{\nu}_{A} - \frac{\rho_{A}}{\rho\eta}\zeta^{\mu}$$

is independent of the choice of U^{μ} . In the L.L. frame $(U^{\mu} = E^{\mu})$ we have $\zeta^{\mu} = 0$ and 3.4.3 says that q^{μ}_{A} is the A^{th} particle flow relative to the momentum.

Next define e and ψ^{μ} by

3.4.10 $$J^{\mu} = \rho e U^{\mu} + \psi^{\mu} \quad ; \quad \psi^{\mu}U_{\mu} = 0 \quad .$$

By considering a frame wherein $U^{\mu} = (0,0,0,1)$ we see that ψ^{μ} is the observed current. It is easy to see that

3.4.11 $$e\rho = \sum_{A} e_{A}\rho_{A} \quad ; \quad \psi^{\mu} = \sum_{A} e_{A}\xi^{\mu}_{A} \quad .$$

Under a transformation $U^\mu = U_*^\mu + \lambda^\mu$ we have

$$\psi_*^\mu = \psi^\mu + \rho e\lambda^\mu$$

3.4.12

$$\zeta_*^\mu = \zeta^\mu + (\mu+p)\lambda^\mu$$

Thus the quantity

3.4.13 $$j^\mu = \psi^\mu - \frac{e}{\eta}\zeta^\mu = \sum_A e_A(q_A^\mu)$$

is independent of U^μ (to within $O\epsilon^2$) . In the L-L frame $(U^\mu = E^\mu)$ we have $\zeta^\mu = 0$, so j^μ is the current relative to the momentum.

The electro magnetic terms of 3.3.18 are

$$-\frac{1}{T}\left(\frac{F^{\mu\nu}J_\nu \Delta_\mu^\lambda \zeta_\lambda}{\eta\rho} + F^{\mu\nu}J_\nu U_\mu\right) .$$

In view of 3.4.4, 3.4.5, 3.4.7 and the skew symmetry of $F^{\mu\nu}$ we have

3.4.14 $$-\frac{1}{T}\left(\frac{F^{\mu\nu}J_\nu \Delta_\mu^\lambda \zeta_\lambda}{\eta\rho} + F^{\mu\nu}J_\nu U_\mu\right) = -\frac{1}{T}\left(F^{\mu\nu}J_\nu\left(\frac{1}{\eta\rho}\zeta_\mu + U_\mu\right)\right.$$

$$= -\frac{1}{T}(F^{\mu\nu}j_\nu U_\mu) + \left.\frac{F^{\lambda\nu}\Sigma\rho_k\xi_k^\nu}{\eta\rho}\zeta_\lambda\right) .$$

The second term is quadratic in the small fluxes ξ_k^ν, ζ_λ . We will therefore put

3.4.15 $$A^{\mu} = \delta^{\mu} - \frac{1}{T} F^{\lambda\nu} \frac{\Sigma e_k \xi_k^{\nu}}{\eta\rho}$$

and rewrite 3.3.18 as

3.4.16 $$S^{\mu}_{,\mu} = \left[\left(\frac{1}{T}\right)_{,\nu} \zeta^{\nu} - \sum_A \theta_{A,\nu} \xi_A^{\nu} + \frac{P_{,\nu}}{\eta\rho T} \zeta^{\nu} \right]$$

$$+ A^{\mu} \zeta_{\mu} - \frac{1}{T} \tau^{\lambda\mu} \epsilon_{\lambda\mu} - \sum \theta_A M^{\nu}_{A,\nu} - \frac{1}{T} F^{\mu\nu} j_{\nu} U_{\mu} \quad .$$

We may rewrite the quantity in square brackets in terms of q_A^{ν} :

$$\left(\frac{1}{T}\right)_{,\nu} \zeta^{\nu} - \sum_A \theta_{A,\nu} \xi_A^{\nu} + \frac{P_{,\nu}}{\eta\rho T} \zeta^{\nu} = \zeta^{\nu} \left(\left(\frac{1}{T}\right)_{,\nu} + \frac{P_{,\nu}}{\eta\rho T} + \frac{1}{\eta} \sum_A \theta_A \left(\frac{\rho_A}{\rho}\right)_{,} \right)$$

$$-\zeta^{\nu} \left(\frac{1}{\eta} \sum_A \theta_A \left(\frac{\rho_A}{\rho}\right)_{,\nu} \right) - \sum_A \theta_{A,\nu} \xi_A^{\nu}$$

3.4.17 $$= \frac{\zeta^{\nu} \theta_{,\nu}}{\eta} - \zeta^{\nu} \frac{1}{\eta} \sum \left(\frac{\theta_A \rho_A}{\rho}\right)_{,\nu}$$

$$+ \zeta^{\nu} \frac{1}{\eta} \sum \theta_{A,\nu} \frac{\rho_A}{\rho} - \sum \theta_{A,\nu} \xi_A^{\nu}$$

$$= -\sum \theta_{A,\nu} \left(\xi_A^{\nu} - \frac{\rho_A}{\rho_{\eta}} \zeta^{\nu}\right)$$

$$= -\sum \theta_{A,\nu} q_A^{\nu} \quad .$$

Here we have used equations 3.2.5, 3.2.4, 3.4.3. We may now write $S^{\mu}_{,\mu}$ as

3.4.18
$$S^{\mu}_{,\mu} = -\sum \theta_{A,\nu}\, q^{\mu}_{A} - \frac{1}{T}\tau^{\lambda\mu}\epsilon_{\lambda\mu} - \sum \theta_{A}\, M^{\nu}_{A,\nu}$$
$$- \frac{1}{T} F^{\lambda\mu} J_{\mu} U_{\lambda} + A^{\mu}\zeta_{\mu} \quad .$$

If we assume the coefficients of viscosity, heat conduction, electrical conductivity and diffusion as small, we may drop $A^{\mu}\zeta_{\mu}$. In view of 3.4.7 we then have

3.4.19
$$S^{\mu}_{,\mu} = -\frac{1}{T}\sum_{A} T\theta_{A}\, M^{\nu}_{A,\nu} - \frac{1}{T}\sum_{A} (T\theta_{A,\nu} + \epsilon_{A} F^{\lambda\cdot}_{\nu} U_{\lambda}) q^{\nu}_{A}$$
$$- \frac{1}{T}\tau^{\lambda\mu}\epsilon_{\lambda\mu} \quad .$$

The terms in 3.4.13 may be interpreted as follows:

(a) $-\sum \theta_{A}\, M^{\nu}_{A,\nu}$ is the entropy production due to the chemical reaction.

(b) $-\sum_{A} (T\theta_{A,\nu} + e_{A} F^{\lambda\cdot}_{\nu} U_{\lambda}) q^{\nu}_{A}$ is the entropy production due to diffusion (diffusion of charged components = current flow).

(c) $-\frac{1}{T}\tau^{\lambda\mu}\epsilon_{\lambda\mu}$ is the entropy production due to viscous forces.

If we assume a single reaction we may write

3.4.20 $$M^{\nu}_{A,\nu} = \nu_A V \quad ,$$

where V is the velocity of the reaction. Putting

3.4.21 $$\varphi = \sum_A T\theta_A \nu_A$$

we have

3.4.22 $$\sum_A T\theta_A M^{\nu}_{A,\nu} = \varphi V \quad .$$

To separate bulk viscosity effects from ordinary viscosity we may put

3.4.23 $$\pi = \frac{1}{3} \tau^{\alpha}_{\alpha} \quad ; \quad \bar{\tau}^{\alpha\beta} = \tau^{\alpha\beta} - \pi\Delta^{\alpha\beta} \quad .$$

The expression for $S^{\alpha}_{,\alpha}$ may then be written

3.4.24 $$S^{\mu}_{,\mu} = \frac{-1}{T}\left\{\varphi V + \sum_A (T\theta_{A,\nu} + \rho_A F^{\lambda \cdot}_{\nu} U_{\lambda}) q^{\nu}_A + \bar{\tau}^{\alpha\beta}\epsilon_{\alpha\beta} + \pi\epsilon^{\alpha}_{\alpha}\right\}$$

The terms in $S^{\mu}_{,\mu}$ are of 3 kinds: products of scalars, products of vectors and products of tensors. If we assume that the system is isotropic, the terms of different species must be independent.

Section 5: Conclusions and Interpretation.

A. Let us assume that V and π are linearly dependent on φ and $\epsilon^{\alpha}_{\alpha}$. Then

3.5.1
$$V = a\varphi + b\epsilon^{\alpha}_{\alpha}$$
$$\pi = c\epsilon^{\alpha}_{\alpha} + d\varphi$$

The contribution of these scalar quantities to the entropy production is then

3.5.2
$$-\frac{1}{T}\{a\varphi^2 + b\epsilon^{\alpha}_{\alpha}\varphi + d\epsilon^{\alpha}_{\alpha}\varphi + c(\epsilon^{\alpha}_{\alpha})^2\} .$$

The requirement that $S^{\alpha}_{\alpha} \geq 0$ then implies that

$$-\{a\alpha^2 + (b+d)\alpha\beta + c\beta^2\}$$

be positive definite.

B. Let us assume that $\underset{A}{q}_{\nu}$ is a linear function of $T\theta_{\beta,\nu} + \epsilon_{\beta}F^{\lambda\cdot}_{\ \nu}U_{\lambda}$. Then we have

3.5.3
$$\underset{A}{q}_{\nu} = \sum_{B} (L_{AB})(T\theta_{B,\nu} + e_B F^{\lambda\cdot}_{\ \nu}U_{\lambda}) .$$

Let us introduce the diffusion flux D^{μ}_{A} of the fluid A relative to the total mass flux M^{μ} :

3.5.4
$$D^{\mu}_{A} = M^{\mu}_{A} - \frac{\rho_A}{\rho} M^{\mu} .$$

Since, by definition

3.5.5
$$q^{\mu}_{A} = \xi^{\mu}_{A} - \frac{\rho_A}{\rho_{\eta}} \zeta^{\mu}$$

we may put

3.5.6 $$D^{\mu}_{A} = q^{\mu}_{A} - \frac{\rho_A}{\rho_\eta} (\xi^{\mu} - \frac{1}{\eta} \zeta^{\mu}) .$$

Summation of 3.5.5 yields

3.5.7 $$\sum_{A} q^{\mu}_{A} = \xi^{\mu} - \frac{1}{\eta} \zeta^{\mu} .$$

We may then combine 3.5.3, 3.5.6 and 3.5.7 to get

3.5.8 $$D_{A^{\mu}} = \sum_{B} \left\{ L_{AB} - \frac{\rho_A}{\rho} \sum_{C} L_{CB} \right\} \left\{ T\theta_{B,\mu} + e_B F^{\lambda\cdot}_{\mu} U_{\lambda} \right\} .$$

If we define N_{AB} by

3.5.9 $$N_{AB} = L_{AB} - \frac{\rho_A}{\rho} \sum_{C} L_{CB}$$

we get

3.5.10 $$D_{A^{\mu}} = \sum_{B} N_{AB} (T\theta_{B,\mu} + e_B F^{\lambda\cdot}_{\mu} U_{\lambda}) .$$

If no magnetic field is present we get

3.5.11 $$D_{A^{\mu}} = \sum N_{AB} T\theta_{B,\mu}$$

which is the relativistic generalization of Fick's law of diffusion, including the possibility of cross effect between fluids of different kinds.

Next let us see what 3.5.3 implies about the current vector. If we take the vector $U^{\alpha} = E^{\alpha}$, j^{μ} is the observed current. We have

3.5.12 $$j_{\mu} = \sum_A e_A q_A^{\mu}$$

$$= \sum_A e_A \sum_B L_{AB}(T\theta_{B,\mu} + e_B F^{\lambda}{}_{\mu} U_{\lambda})$$

3.5.13 $$j_{\mu} = \sum_A \sum_B e_A L_{AB} T\theta_{B,\mu} + \sigma F^{\lambda}{}_{\mu} U_{\lambda}$$

where

3.5.14 $$\sigma = \sum_A e_A \sum_B L_{AB}\, e_B \quad .$$

For a mixture in chemical equilibrium $(\theta_{B,\mu} = 0)$ we have the relativistic Ohms law

3.5.15 $$j_{\mu} = \sigma F^{\lambda}{}_{\mu} U_{\lambda} \quad .$$

In a mixture not in chemical equilibrium 3.5.13 shows that the current depends on the gradients of the chemical potentials as well as the electric field.

C. Let us assume that $\bar{\tau}^{\alpha\beta}$ is a linear function of $\epsilon_{\alpha\beta}$. Isotropy then leads us to

3.5.16 $$\bar{\tau}^{\alpha\beta} = n \, \Delta^{\alpha\gamma} \, \Delta^{\beta\delta} \{U_{\gamma,\delta} + U_{\delta,\gamma}\} - \frac{2}{3} n \, \Delta^{\alpha\beta} \, U^{\gamma}_{,\gamma} \quad .$$

In view of 3.4.17, 3.5.1 we may put

3.5.17 $$\tau^{\alpha\beta} = n \, \Delta^{\alpha\gamma} \, \Delta^{\beta\delta} \{U_{\gamma,\delta} + U_{\delta,\gamma}\} + (C - \frac{2}{3} n) \, \Delta^{\alpha\beta} \, U^{\gamma}_{,\gamma} + d\varphi U^{\gamma}_{\gamma} \quad .$$

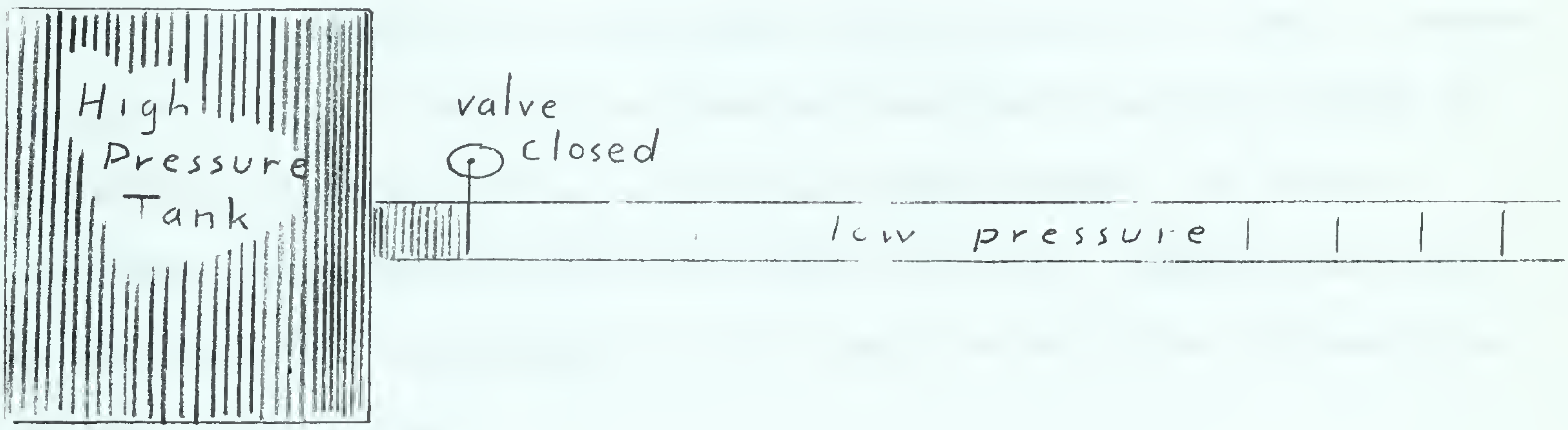
The first two terms are the usual terms in the stress-tensor; the last one expresses the possibility of cross effects between the bulk viscosity and the chemical reaction. We may note here that similar work (with special selection of the 4 velocity U^{α}) on the relativistic irreversible thermodynamics of mixtures has been done by Stueckelberg [2] and Kluitenberg, De Groot and Mazur [8], [9], [10].

CHAPTER IV

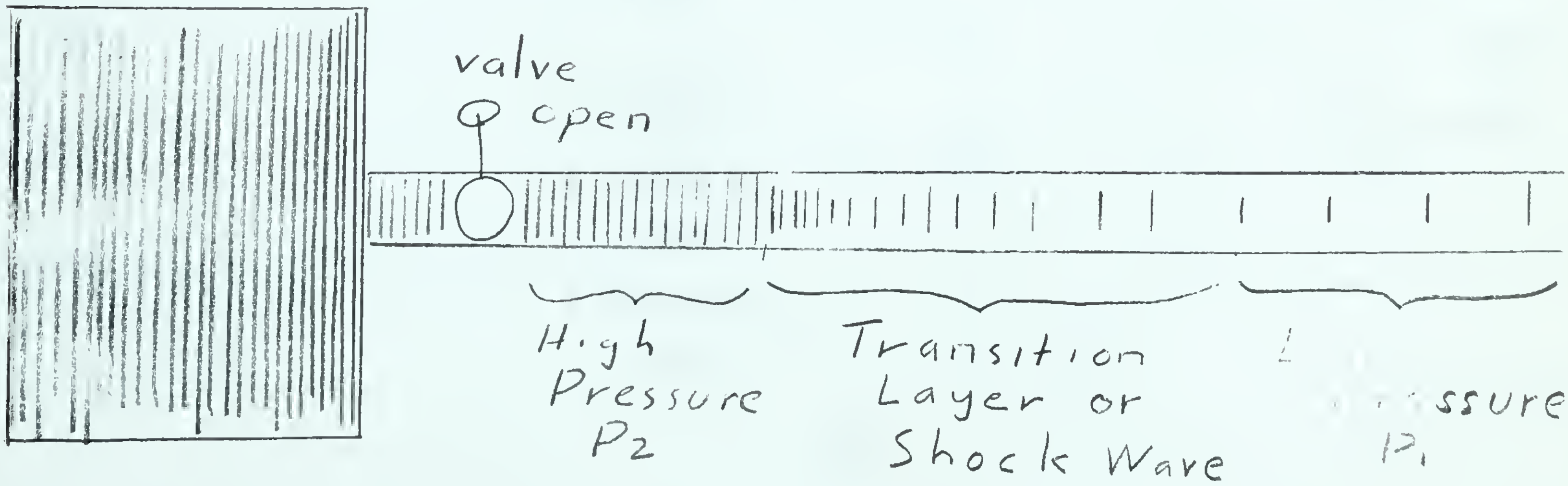
THE STRUCTURE OF PLANE SHOCK WAVES

Section 1: Introduction.

A shock wave is a physical phenomenon which may be produced by an apparatus as shown in the diagram:



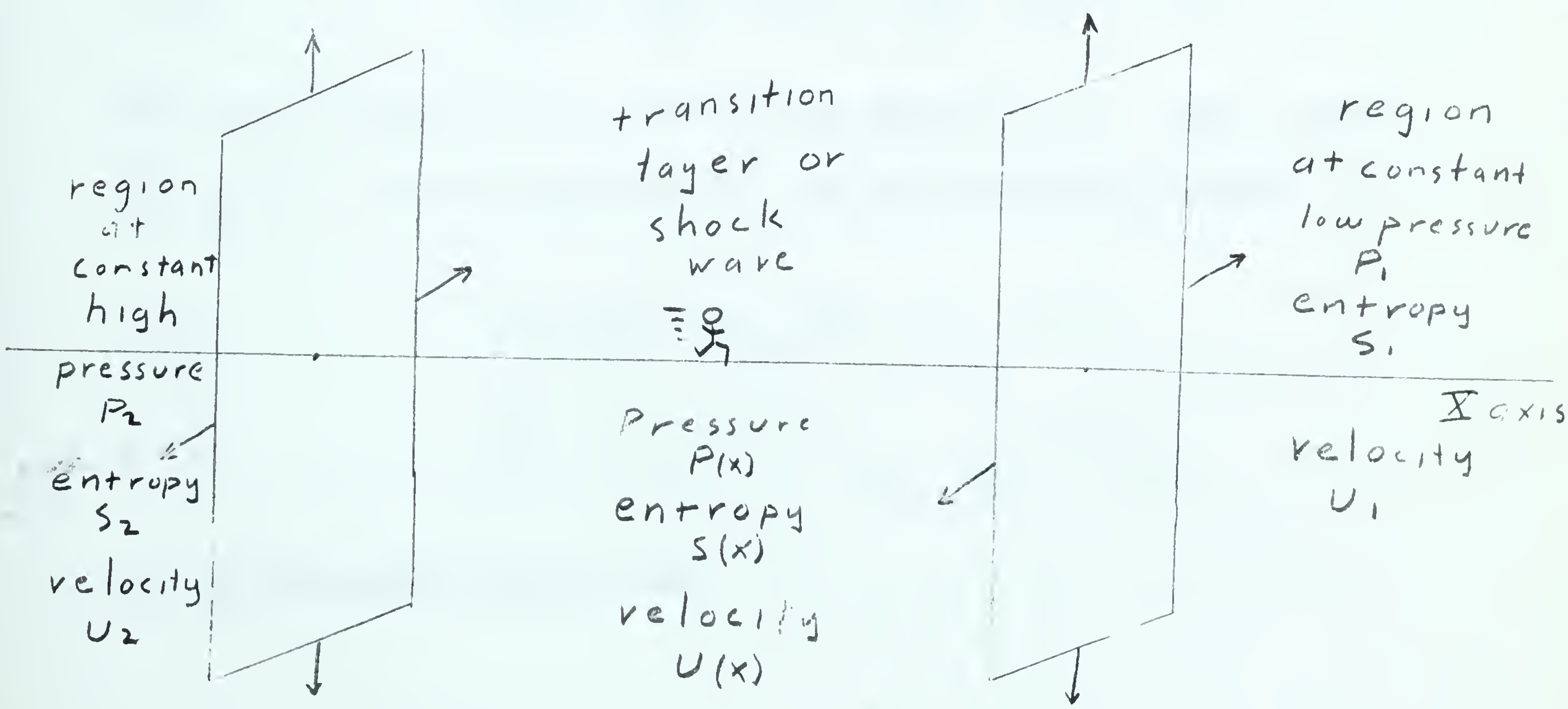
When the valve is opened, fluid in the tank rushes into the pipe, causing a phenomena called a shock wave:



If the pipe is long enough there is a uniform region at (high) pressure P_2 , entropy S_2 , velocity U_2 ; a transition layer or shock wave in which P, S, U vary; and a uniform region (which the shock wave has not yet reached) at pressure P_1 , entropy S_1 , velocity U_1 .

A relativistic treatment of shock waves is of current interest. In the theory of gravitational collapse (recently proposed by Hoyle and Fowler to account for quasi-stellar radio sources) the shocks generated by novae and supernovae should certainly be described relativistically.

Our purpose is to investigate the variation of P and U across the shock wave. To make the mathematics practicable we will consider an infinite plane shock wave in an otherwise uniform medium. We adopt a coordinate system wherein the shock layer is at rest. Assuming the plane of the shock wave perpendicular to the X axis, we will have all quantities as function of X alone.



The observer runs along with the shock wave, so that the quantities will not depend on time.

Section 2: The Conservation Equations.

For one dimensional flow (steady in the frame wherein the shock wave is at rest) we have $(x^1 = x)$

4.2.1 $$\frac{\partial M^1}{\partial x} = 0 \quad ; \quad \frac{\partial T^{\mu 1}}{\partial x} = 0$$

with first integrals

4.2.2 $$M^1 = J \quad ; \quad T^{\mu 1} = K^\mu$$

where J and K^μ are constant. Writing these out explicitly

4.2.3 $$\rho U^1 + \xi^1 = J$$

4.2.4 $$\eta\rho U^\mu U^1 + Pg^{\mu 1} + U^\mu \zeta^1 + U^1 \zeta^\mu + \tau^{\mu 1} = K^\mu \quad .$$

The obvious choice of U^μ is that which makes $\xi^1 = 0$. Thus, select $U^\mu = \frac{1}{\rho} M^\mu$. For this choice of U^μ , as we have shown in Chapter II

4.2.5 $$q^\mu = \frac{-1}{\eta} \zeta^\mu = -K\theta_{,\beta} \Delta^{\mu\beta} \quad ; \quad \xi^\mu = 0$$

4.2.6 $$\tau^{\mu\nu} = -\lambda_1 \Delta^{\alpha\mu} \Delta^{\beta\nu} \epsilon_{\alpha\beta} - \lambda_2 \Delta^{\mu\nu} \Delta^{\alpha\beta} \epsilon_{\alpha\beta} \quad .$$

For one dimensional flow we have

$$U^1 = U \qquad U^3 = 0$$

4.2.7

$$U^2 = 0 \qquad U^4 = \sqrt{(1 + U^2)}$$

Substituting these into 4.2.5 and 4.2.6 then yields

4.2.8 $$\zeta^1 = \eta K(1+U^2)\frac{\partial\theta}{\partial x} \quad ; \quad \zeta^4 = \eta KU\sqrt{(1+U^2)}\frac{\partial\theta}{\partial x}$$

$$\tau^{11} = -(\lambda_1+\lambda_2)(1+U^2)\ \frac{\partial U}{\partial x}$$

4.2.9

$$\tau^{41} = -(\lambda_1+\lambda_2)U\ \sqrt{1+U^2}\ \frac{\partial U}{\partial x}$$

If we simplify by putting

4.2.10 $$a(x) = \eta K \qquad b(x) = \lambda_1+\lambda_2$$

we get, upon substitution into 4.2.3 and 4.2.4 the three equations:

4.2.11 $$\rho U = J$$

4.2.12 $$P + \eta\rho U^2 + 2aU(1+U^2)\ \frac{\partial\theta}{\partial x} - b(1+U^2)\ \frac{\partial U}{\partial x} = K^1$$

4.2.13 $$\eta\rho U\ \sqrt{1+U^2} + a(1+2U^2)\ \sqrt{1+U^2}\ \frac{\partial\theta}{\partial x} - bU\ \sqrt{1+U^2}\ \frac{\partial U}{\partial x} = K^2 \quad .$$

These equations must hold in the undisturbed region 1 ahead of the shockwave, wherein $\frac{\partial\theta}{\partial x} = \frac{\partial U}{\partial x} = 0$. Hence, denoting quantities

measured in this region by 1 we have

4.2.14 $$J = \rho_1 U_1$$

4.2.15 $$K^1 = \eta_1 \rho_1 U_1^2$$

4.2.16 $$K^2 = \eta_1 \rho_1 U_1 \sqrt{1+U_1^2} \quad .$$

The final form of the equations to be solved for $P(x)$ and $S(x)$ is then

4.2.17 $$\rho U = J$$

4.2.18 $$P - P_1 + \left(\frac{\eta}{\rho} - \frac{\eta_1}{\rho_1}\right) J^2 + 2aU(1+U^2)\frac{\partial\theta}{\partial x} - b(1+U^2)\frac{\partial U}{\partial x} = 0$$

4.2.19 $$\eta\sqrt{1+U^2} - \eta_1\sqrt{1+U_1^2} + a(1+2U^2)\sqrt{1+U^2}\frac{\partial\theta}{\partial x} - bU\sqrt{1+U^2}\frac{\partial U}{\partial x} = 0 \quad .$$

Section 3: The Relativistic Rankine-Hugoniot Equations.

The equations 4.2.17 - 19 hold in the portion of the fluid behind the shockwave (2) . In this region $\frac{\partial\theta}{\partial x} = \frac{\partial U}{\partial x} = 0$. Thus, denoting the quantities measured in region 2 by a subscript 2 we have

4.3.20 $$\rho_1 U_1 = \rho_2 U_2$$

4.3.21 $$\left(\frac{\eta_2}{\rho_2} - \frac{\eta_1}{\rho_1}\right) J^2 + (P_2 - P_1) = 0$$

4.3.22 $$\eta_2\sqrt{1 + U_2^2} - \eta_1\sqrt{1 + U_1^2} = 0 \quad .$$

Multiply 4.3.21 by $\frac{\eta_2}{\rho_2} + \frac{\eta_1}{\rho_1}$; multiply 4.3.22 by $\eta_2 \sqrt{1+U_2^2} + \eta_1 \sqrt{1+U_1^2}$ and subtract. The result is

4.3.23 $$(\eta_2^2 - \eta_1^2) - (P_2 - P_1)\left(\frac{\eta_2}{\rho_2} + \frac{\eta_1}{\rho_1}\right) = 0 .$$

The equations 4.3.20, 4.3.21 and 4.3.23 are the Relativistic Rankin-Hugoniot equations (cf. Taub [4]). The equation 4.3.23 implies an interesting relation between the pressure differential $P_2 - P_1$ and the entropy differential $S_2 - S_1$. We will expand $\eta^2 - \eta_1^2$ and $\frac{\eta}{\rho} + \frac{\eta_1}{\rho_1}$ in powers of $S - S_1$ and $P - P_1$ about a point in the region 1. Noting the thermodynamic relations

4.3.24 $$V = \frac{1}{\rho} \quad ; \quad \left(\frac{\partial(\eta/\rho)}{\partial S}\right)_P = VT + \eta\left(\frac{\partial V}{\partial S}\right)_P$$

$$\left(\frac{\partial\, \eta/\rho}{\partial P}\right)_S = V^2 + \eta\left(\frac{\partial V}{\partial P}\right)_S$$

$$\left(\frac{\partial^2 \eta/\rho}{\partial P^2}\right)_S = 3V\left(\frac{\partial V}{\partial P}\right)_S + \eta\left(\frac{\partial^2 V}{\partial P^2}\right)_S$$

we may write

4.3.25 $$\eta - \eta_1 = T_1(S-S_1) + V_1(P-P_1) + \left(\frac{\partial V}{\partial P}\right)_{S_1} \frac{(P-P_1)^2}{2!}$$

$$+ \left(\frac{\partial^2 V}{\partial P^2}\right)_{S_1} \frac{(P-P_1)^3}{3!} + o\,(S-S_1) + o\,(P-P_1)^4$$

4.3.26
$$\frac{\eta}{\rho} + \frac{\eta_1}{\rho_1} = 2\,\frac{\eta_1}{\rho_1} + \left[V_1 T + \eta_1 \left(\frac{\partial V}{\partial S}\right)_{P_1}\right](S-S_1)$$

$$+ \left[V_1^2 + \eta_1 \left(\frac{\partial V}{\partial P}\right)_{S_1}\right](P-P_1)$$

$$+ \left(\frac{\partial^2 \eta/\rho}{\partial P^2}\right)_{S_1} \frac{(P-P_1)^2}{2!} + o(S-S_1) + o(P-P_1)^3 \;.$$

Note that in 4.3.25 and 4.3.26, $o(S-S_1)$ is a sum of terms of form $A(S-S_1)^m(P-P_1)^n$; $m,n \geq 1$.

We then have

4.3.27
$$\eta^2-\eta_1^2 - (P-P_1)\left(\frac{\eta}{\rho} + \frac{\eta_1}{\rho_1}\right) = 2\eta_1 T_1(S-S_1) - \left(\frac{\partial^2(\eta/\rho)}{\partial P^2}\right)_{S_1} \frac{(P-P_1)^3}{3!}$$

$$+ o(S-S_1) + o(P-P_1)^4 \;.$$

In view of equation 4.3.23, we must conclude that

4.3.28
$$S_2-S_1 = \frac{1}{12\eta_1 T_1}\left(\frac{\partial^2(\eta/\rho)}{\partial P^2}\right)_{S_1} (P_2-P_1)^3$$

i.e., the total entropy discontinuity is of the third order in the total pressure discontinuity.

Section 4: Approximation to the differential equations.

In this section we will derive an approximation for the equations

4.2.17 - 4.2.19 in a form suitable for solution. Let us define ϵ by

4.4.1 $$\epsilon = \frac{P_2 - P_1}{P_1} \quad .$$

Let us assume that the major part of the variation in P takes place over a distance of order L . We shall assume that

4.4.2 $$\epsilon \to 0 \Longleftrightarrow L \to \infty \quad .$$

For out approximation we shall retain terms of order ϵ , $\frac{1}{L}$, ϵ^2 , $\frac{1}{L^2}$, $\frac{\epsilon}{L}$ and drop all higher ones. We then have, assuming that P varies smoothly and monotonically through the shock wave

4.4.3 $$P - P_1 = O(\epsilon)$$
$$\frac{dP}{dx} \sim \frac{P_2 - P_1}{L} = O(\frac{\epsilon}{L}) \quad .$$

Assuming the variation of S inside the shockwave is smooth and subject to only a few maxima and minima we have

4.4.4 $$\left|\frac{dS}{dx}\right|_{max} \sim \frac{|S-S_1|_{max}}{L} \quad .$$

Let us now begin by multiplying 4.2.18 by $\eta/\rho + \eta_1/\rho_1$; multiplying 4.2.19 by $(\eta\sqrt{1+U^2} + \eta_1\sqrt{1+U_1^2})/J$; and subtracting. The result is

4.4.5 $$\eta^2 - \eta_1^2 - (P-P_1)(\eta/\rho + \eta_1/\rho_1) + 2(\eta_1/J)(1+U_1^2)a_1 \frac{d\theta}{dx} + o(S-S_1) = 0 \quad .$$

In view of 4.3.27 we have

4.4.6
$$S-S_1 + o(S-S_1) = \frac{-(1+U_1^2)}{J} \frac{a_1}{T_1} \left(\frac{\partial \theta}{\partial P} \right)_{S_1} \frac{dP}{dx}$$

$$\frac{-(1+U_1^2)}{J} \frac{a_1}{T_1} \left(\frac{\partial \theta}{\partial S} \right)_{P_1} \frac{dS}{dx} \quad .$$

Using 4.4.4 we have

4.4.7
$$|S-S_1|_{max} \sim (const) \left|\frac{dP}{dx}\right|_{max} = O(\epsilon/L) \quad .$$

Thus, by 4.4.4 we have

4.4.8
$$\left|\frac{dS}{dx}\right| = O\left(\frac{\epsilon}{L^2}\right) \quad .$$

To our degree of approximation, we may therefore drop $o(S-S_1)$ and $\frac{dS}{dx}$. Equation 4.4.6 then becomes

4.4.9
$$S-S_1 = -\frac{1+U_1^2}{J} \frac{a_1}{T_1} \left(\frac{\partial \theta}{\partial P} \right)_{S_1} \frac{dP}{dx} \quad .$$

Next let us approximate 4.2.18. Using 4.3.26 we get, to the desired degree of approximation,

4.4.10 $$J^2\left\{\left(\frac{\partial\ \eta/\rho}{\partial S}\right)_{P_1}(S-S_1) + \left(\frac{\partial\ \eta/\rho}{\partial P}\right)_{S_1}(P-P_1) + \left(\frac{\partial^2\ \eta/\rho}{\partial P^2}\right)_{S_1}\frac{(P-P_1)^2}{2!}\right\}$$

$$+ (P-P_1) + 2a_1U_1\left(\frac{\partial\theta}{\partial P}\right)_{S_1} - b_1J\left(\frac{\partial V}{\partial P}\right)_{S_1}(1+U^2)\frac{dP}{dx} = 0 .$$

Note that we have used the relation

4.4.11 $$\frac{dU}{dx} = J\frac{dV}{dx} = J\left(\frac{\partial V}{\partial P}\right)_{S_1}\frac{dP}{dx} .$$

To get the equations in a soluble form let us now substitute 4.4.5 into 4.4.6. The result is

4.4.12 $$\frac{dP}{dx}\left\{-\left(\frac{\partial\ (\eta/\rho)}{\partial S}\right)_{P_1}\frac{a_1}{T_1}\left(\frac{\partial\theta}{\partial P}\right)_{S_1} J + 2a_1U_1\left(\frac{\partial\theta}{\partial P}\right)_{S_1} - Jb_1\left(\frac{\partial V}{\partial P}\right)_{S_1}\right\}$$

$$= \frac{-J^2}{1+U_1^2}\left\{\left[\left(\frac{\partial\ \eta/\rho}{\partial P}\right)_{S_1} + \frac{1}{J^2}\right](P-P_1) - \left(\frac{\partial^2\ \eta/\rho}{\partial P^2}\right)_{S_1}\frac{(P-P_1)^2}{2!}\right\} .$$

This may be simplified by observing that the right hand side is a polynomial of degree 2 in P with the zeros P_1 and P_2 (left hand side vanishes in region 1 and region 2). Thus

4.4.13 $$\frac{dP}{dx} = \frac{\frac{-J^2}{2}\left(\frac{\partial^2\ \eta/\rho}{\partial P^2}\right)_{S_1}(P-P_1)(P-P_2)}{(1+U_1^2)\left[-\left(\frac{\partial(\eta/\rho)}{\partial S}\right)_{P_1}\frac{a_1}{T_1}\left(\frac{\partial\theta}{\partial P}\right)_{S_1} J + 2a_1U_1\left(\frac{\partial\theta}{\partial P}\right)_{S_1} - Jb_1\left(\frac{\partial V}{\partial P}\right)_{S_1}\right]}$$

In view of the thermodynamic identities

$$\left(\frac{\partial(\eta/\rho)}{\partial S}\right)_{P_1} = V_1 T_1 + \frac{\eta_1}{T_1^2}\left(\frac{\partial V}{\partial S}\right)_{P_1}$$

$$\left(\frac{\partial\theta}{\partial P}\right)_{S_1} = V_1 - \frac{\eta_1}{T_1}\left(\frac{\partial V}{\partial S}\right)_{P_1}$$

this may be rewritten as

4.4.14 $$\frac{dP}{dx} = \frac{-\frac{J}{2}\left(\frac{\partial^2(\eta/\rho)}{\partial P^2}\right)_{S_1}(P-P_1)(P-P_2)}{(1+U_1^2)\left[a_1 T_1\left(\frac{\partial\theta}{\partial P}\right)_{S_1}^2 - b_1\left(\frac{\partial V}{\partial P}\right)_{S_1}\right]} .$$

It is interesting to compare this with the classical expression for $\frac{dP}{dx}$ as given by Landau-Lifshitz [3]

$$\frac{dP}{dx} = \frac{-\frac{J}{2}\left(\frac{\partial^2 V}{\partial P^2}\right)_{S_1}(P-P_1)(P-P_2)}{\left[\frac{K_1}{T_1}\left(\frac{\partial T}{\partial P}\right)_{S_1}^2 - \left(\frac{4}{3}\eta_1 + \zeta_1\right)\left(\frac{\partial V}{\partial P}\right)_{S_1}\right]} .$$

The expressions are very similar, the only difference being the replacement of V by $(\frac{\eta}{\rho})$ and of T by θ . The replacement of V by $(\frac{\eta}{\rho})$ has already been commented on (cf. Israel [5]). The term $(\frac{\partial\theta}{\partial P})_{S_1}$ is in the classical approximation equal to $-\frac{1}{T_1^2}(\frac{\partial T}{\partial P})_{S_1}$: as $\theta = \frac{\eta}{T} - \frac{S}{C^2}$, in

the classical limit $n \to 1$ and $c \to \infty$ we have

$$\theta = \left(\frac{1}{T}\right) .$$

Thus

$$a_1 T_1 \left(\frac{\partial \theta}{\partial P} \right)^2_{S_1} = a_1 T_1 \left(\frac{1}{T_1^2} \left(\frac{\partial T}{\partial P} \right)_{S_1} \right)^2$$

$$= \frac{a_1}{T_1^3} \left(\frac{\partial T}{\partial P} \right)^2_{S_1}$$

$$= K_1 \left(\frac{\partial T}{\partial P} \right)^2_{S_1} .$$

Thus our equation 4.4.10 reduces, as it should, to the classical expression in the classical limit.

Section 5: Conclusion.

To continue, in equation 4.4.10 put

4.5.1 $$A = \frac{-\frac{J}{2}\left(\frac{\partial^2 (\eta/\rho)}{\partial P^2} \right)_{S_1}}{(1 + U_1^2)\left[a_1 T \left(\frac{\partial \theta}{\partial P} \right)^2_{S_1} - b_1 \left(\frac{\partial V}{\partial P} \right)_{S_1} \right]} .$$

We then have

4.5.2 $$\frac{dP}{dx} = A(P - P_1)(P - P_2)$$

or

4.5.3 $$\frac{dP}{(P-P_1)(P-P_2)} = A dx \quad .$$

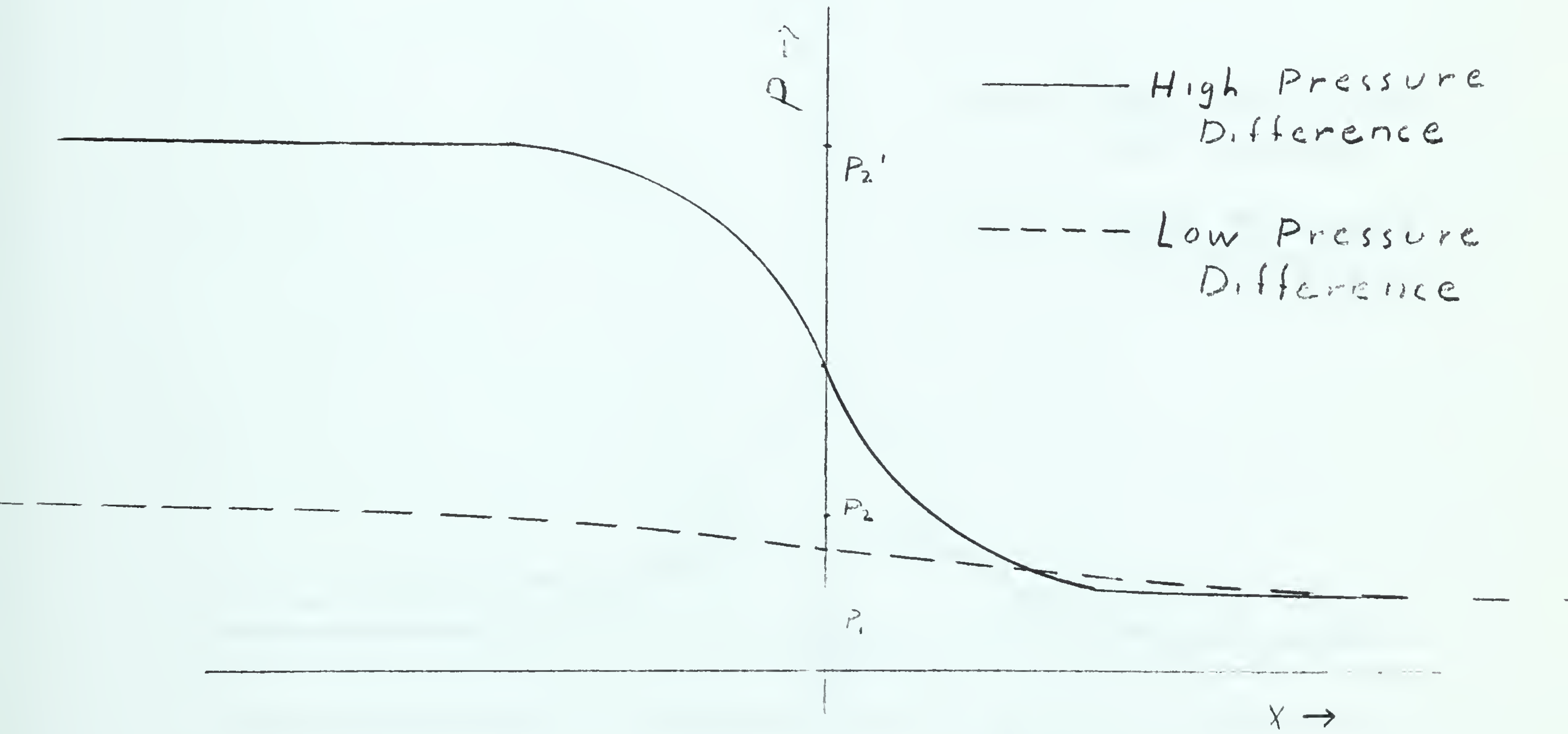
Integrating, and putting $x = 0$ where $P = \frac{1}{2}(P_2 + P_1)$ we have

$$Ax = \frac{1}{\frac{1}{2}(P_2-P_1)} \tanh^{-1} \frac{P-\frac{1}{2}(P_2+P_1)}{\frac{1}{2}(P_2-P_1)}$$

or,

4.5.4 $$P = \tfrac{1}{2}(P_2 + P_1) + \tfrac{1}{2}(P_2 - P_1) \tanh(\tfrac{1}{2}(P_2-P_1)Ax) \quad .$$

The consequences of 4.5.4 are best seen in a graph.

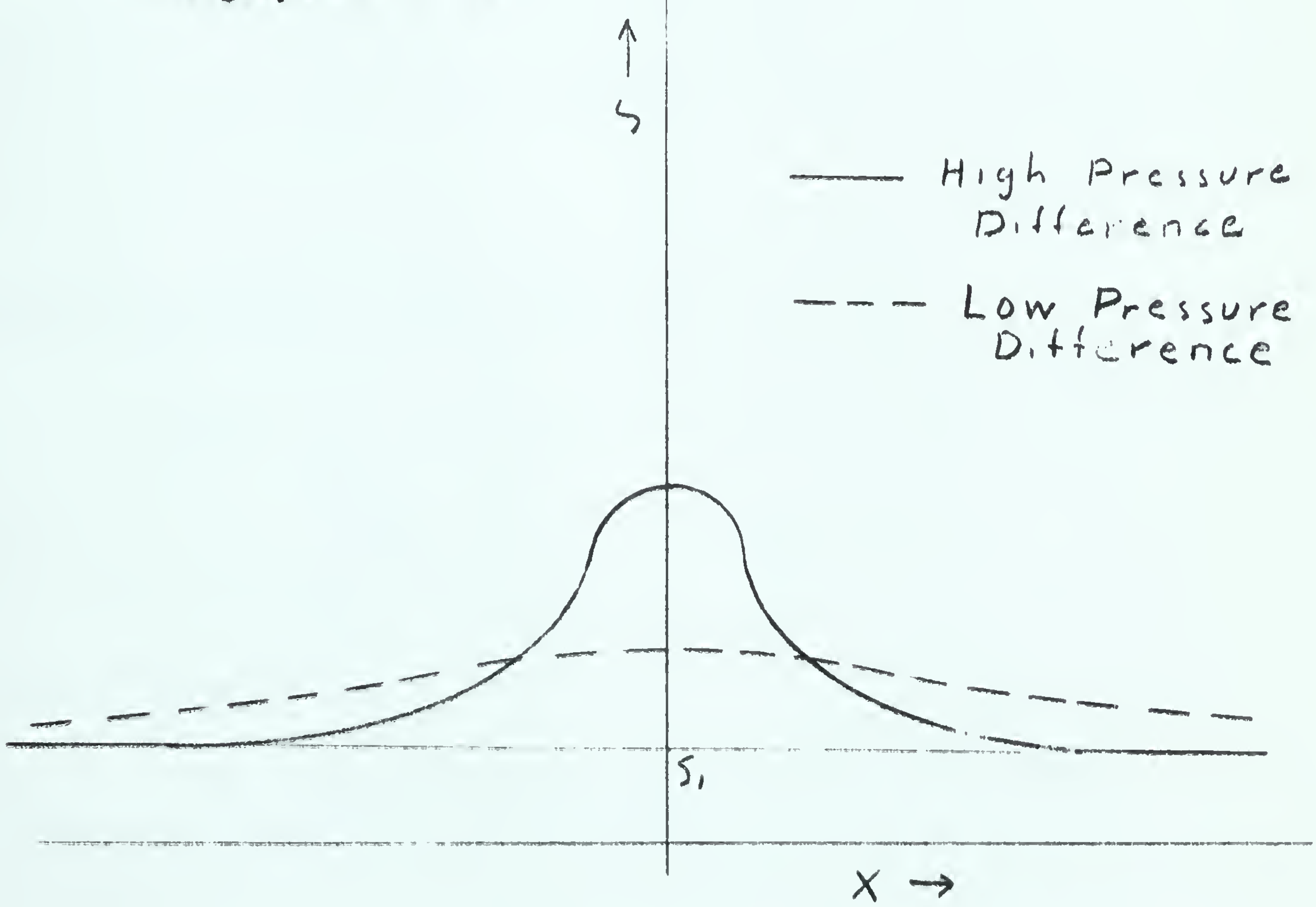


As the diagram shows, the higher the pressure differential, the more sharply defined and narrow is the region which could be called a "shock wave".

The equations 4.5.4 and 4.5.6 may be combined to get an expression for $S - S_1$:

$$4.5.5 \qquad S - S_1 = \frac{a_1}{8T_1} \frac{\left(\frac{\partial\theta}{\partial P}\right)_{S_1}\left(\frac{\partial^2(\eta/\rho)}{\partial P^2}\right)_{S_1}(P_2-P_1)^2}{\left[a_1T_1\left(\frac{\partial\theta}{\partial P}\right)_{S_1} - b_1\left(\frac{\partial V}{\partial P}\right)_{S_1}\right]} \operatorname{sech}^2\left(\tfrac{1}{2}(P_2-P_1)Ax\right) .$$

The graph of S is



As the graph shows the entropy has a maximum inside the shock wave. 4.5.5 implies that $S_2 - S_1 = 0$. The reason for this, of course, is that $S_2 - S_1$ is of third order in the pressure differential, while we have only retained terms of order 2 in the pressure differential. Hence, to our degree of approximation, $S_2 - S_1$ is in fact 0.

The length L over which the majority of the change in pressure takes place is, according to 4.5.1 and 4.5.5, proportional to $[\frac{1}{2}(P_2-P_1)A]^{-1}$. The A is a constant depending only on the thermodynamic properties of the undisturbed fluid. Hence the length L is proportional to $\frac{1}{\epsilon}$, and our approximation procedure is seen to be merely that of ignoring quantities of order ϵ^3 .

CHAPTER V

TOLMAN'S THEOREM

Section 1: Introduction.

In this chapter we will consider the following problem: in a particular coordinate system (*) the gravitational field appears static. What may be said about the temperature? Tolman [6] has shown that if thermal equilibrium is assumed then

5.1.1 $$\sqrt{g_{44}}\, T = \text{constant} .$$

We will show that this assumption is unnecessary. All that is needed is the reasonable assumption that as $t \to \infty$ all heat flow ceases. We are then enabled to prove that thermal equilibrium is implied by the fact that the field is static.

Section 2: The Form of the Energy Momentum Tensor.

In the starred coordinate system, the field is static. Thus

5.2.1 $$\frac{\partial g_{\mu\nu}}{\partial x^4} \overset{*}{=} 0 \quad ; \quad g_{4i} \overset{*}{=} 0 .$$

Define a vector U^α by

5.2.2 $$U^\alpha \overset{*}{=} (0,0,0,\frac{1}{\sqrt{-g_{44}}}) .$$

It is obvious that

5.2.3 $$U^{\alpha}U_{\alpha} = -1 .$$

We then have the result

5.2.4 $$U^{\alpha}_{\ ,\beta}\, U_{\alpha} = 0 .$$

Also

5.2.5 $$\dot{U}^{\alpha}U_{\alpha} = U^{\alpha}_{\ ,\beta}U^{\beta}U_{\alpha} = 0 .$$

Observe that

5.2.6 $$U_{\alpha,\beta} + \dot{U}_{\alpha}U_{\beta} = \frac{\partial U_{\alpha}}{\partial x^{\beta}} - \left\{ {\rho \atop \alpha\beta} \right\} U_{\rho} + \left[\frac{\partial U_{\alpha}}{\partial x^{\rho}} - \left\{ {\mu \atop \alpha\rho} \right\} U_{\mu} \right] U^{\rho}U_{\beta}$$

or

5.2.7 $$U_{\alpha,\beta} + \dot{U}_{\alpha}U_{\beta} \overset{*}{=} \frac{\partial U_{\alpha}}{\partial x^{\beta}} - \left\{ {4 \atop \alpha\beta} \right\} U_{4} + \left[\frac{\partial U_{\alpha}}{\partial x^{4}} - \left\{ {4 \atop \alpha 4} \right\} U_{4} \right] U^{4}U_{\beta}$$

In view of 5.2.1 this gives

5.2.7 $$U_{\alpha,\beta} + \dot{U}_{\alpha}U_{\beta} = 0 .$$

Contraction of 5.2.7 gives, with 5.2.5

5.2.8 $$U^{\alpha}_{\ ,\alpha} = 0 .$$

Consider the fact that

5.2.9 $$U_{\mu}R^{\mu}_{\ \cdot\alpha\beta\gamma} = U_{\alpha,\beta\gamma} - U_{\alpha,\gamma\beta} .$$

Multiplication by $g^{\alpha\beta}$ gives

5.2.10 $$U_\mu R^\mu_\gamma = U^\beta_{,\beta\gamma} - U^\beta_{,\gamma\beta}$$

$$= U^\beta_{,\gamma\beta} \qquad \text{by 5.2.8}$$

$$= -(\dot{U}^\beta U_\gamma)_{,\beta} \qquad \text{by 5.2.7}$$

$$= -(\dot{U}^\beta_{,\beta})U_\gamma \qquad \text{by 5.2.6 and 5.2.5} \quad .$$

We may then write

5.2.11 $$U_\mu[R^\mu_\gamma - \tfrac{1}{2}\delta^\mu_\gamma R] = (\dot{U}^\alpha_{,\alpha} - \tfrac{1}{2}R)U_\gamma \quad .$$

In view of Einstein's equation

5.2.12 $$\chi T^\alpha_{.\beta} + [R^\alpha_{.\beta} - \tfrac{1}{2}\delta^\alpha_\beta R] = 0$$

5.2.11 enables us to write

5.2.13 $$T^\alpha_{.\beta}U_\alpha = AU_\alpha \quad .$$

Thus we have the result that the velocity U^α of an observer who sees the gravitational field as static is the timelike eigenvector of the energy momentum tensor. According to equations 2.3.8 - 10 and 2.3.14, 2.3.28 we have

5.2.14 $$T^{\mu\nu} = \mu U^{\mu}U^{\nu} + P\,\Delta^{\mu\nu} + \zeta^{\mu}U^{\nu} + \zeta^{\nu}U^{\mu} + \tau^{\mu\nu}$$

where

$$\mu = T^{\mu\nu}U_{\mu}U_{\nu}$$

$$\zeta^{\mu} = -T^{\alpha\beta}\,\Delta^{\mu}_{\alpha}\,U_{\beta}$$

$$\tau^{\mu\nu} = T^{\alpha\beta}\,\Delta^{\mu}_{\alpha}\,\Delta^{\nu}_{\beta} - P\,\Delta^{\mu\nu} \quad .$$

Since U^{μ} is the velocity of the observer in the Landau-Lifshitz frame ($U^{\mu} = E^{\mu}$ = timelike eigenvector of $T^{\mu\nu}$) we may use equation 2.4.10 and conclude that

5.2.15 $$\zeta^{\mu} = 0 \; .$$

If we use equation 2.4.4 in conjunction with 5.2.4, 5.2.5 we get

5.2.16 $$\tau^{\mu\nu} = 0 \; .$$

The energy momentum tensor is now

5.2.17 $$T^{\mu\nu} = (\mu+P)U^{\mu}U^{\nu} + Pg^{\mu\nu} \quad .$$

We know that

$$\frac{\partial T^{\alpha\beta}}{\partial x^{4}} \overset{*}{=} 0$$

in view of 5.2.12 and 5.2.1. Then 5.2.17 and the conservation equation $T^{\alpha\beta}_{\;\;,\beta} = 0$ give

5.2.18 $$\frac{\partial U_\mu}{\partial x^4} \overset{*}{=} \frac{\partial \mu}{\partial x^4} \overset{*}{=} \frac{\partial P}{\partial x^4} = 0 \quad .$$

Since $\rho = \rho(P,\mu)$ we have

5.2.19 $$\frac{\partial \rho}{\partial x^4} \overset{*}{=} 0 \quad .$$

Expressed covariantly 5.2.18 and 5.2.19 are

5.2.20 $$P_{,\mu}U^\mu = \mu_{,\mu}U^\mu = \rho_{,\mu}U^\mu = 0 \quad .$$

The conservation equation $T^{\mu\nu}_{\ \ ,\nu} = 0$ then gives

5.2.21 $$T_{\mu,\nu}^{\ \ \nu} = (\mu+P)\dot{U}_\mu + P_{,\mu} = 0 \quad .$$

Since

$$\begin{aligned}
\dot{U}_\mu &= U_{\mu,\alpha}U^\alpha \\
&\overset{*}{=} U_{\mu,4}U^4 \\
&\overset{*}{=} \frac{-1}{\sqrt{-g_{44}}} \left\{{\alpha \atop \mu 4}\right\} U_\alpha \\
&\overset{*}{=} \frac{1}{g_{44}} [\mu 4, 4] \\
&\overset{*}{=} \frac{1}{g_{44}} \frac{1}{2} \frac{\partial g_{44}}{\partial x_\mu}
\end{aligned}$$

we may write

5.2.22 $$\dot{U}^\mu \overset{*}{=} \frac{\partial \, \ell n\sqrt{-g_{44}}}{\partial x^\mu} \quad .$$

Equation 5.2.21 then gives us

5.2.23 $$- \frac{\partial P}{\partial x^{\mu}} \frac{1}{\mu+P} \stackrel{*}{=} \frac{\partial \ \ell n\sqrt{-g_{44}}}{\partial x^{\mu}} .$$

From the thermodynamic relation 2.3.27, we have

5.2.24 $$\frac{T}{\eta} d\theta = \frac{1}{\mu+P} dP - d(\ell n \ T) .$$

Combining the two above equations gives

5.2.25 $$\frac{T}{\eta} \theta_{,\mu} \stackrel{*}{=} - \frac{\partial}{\partial x_{\mu}} \ell n(\sqrt{-g_{44}} \ T) .$$

Section 3: Proof of Tolman's Theorem.

Multiplication of 5.2.24 by U_{μ} gives, with 5.2.20, the result

5.3.1 $$(\dot{\overline{\sqrt{-g_{44}} \ T}}) = 0 .$$

Now we know that $\sqrt{-g_{44}} \ T$ is constant along the world lines of the observer who see the field as static. If we can show that at some time $\sqrt{-g_{44}} \ T$ is the same for all observers, we know that it is constant throughout space time. To do this we will make the assumption (reasonable in view of the second law of thermodynamics) that at $t = \infty$ thermal equilibrium is reached. Thus at $t = \infty$,

5.3.2 $$\theta_{,\mu} = 0 .$$

Equation 5.2.24 and 5.3.1 then show that $\sqrt{-g_{44}}\, T$ is universally constant. Further 5.2.24 assures us that θ is universally constant. Since the heat flow is dependent on $\theta_{,\mu}$, we have, universally, no heat flow. The system, then, must be in thermal equilibrium. Since

5.3.3 $$\theta = \frac{G}{T} = \text{constant (G = Gibb's Function)}$$

we have

5.3.4 $$G\sqrt{-g_{44}} = \text{constant}$$

which is Klein's theorem. [7]

To sum up: we have shown that the assumption of a static field, plus the assumption of thermal equilibrium at $t = \infty$ are sufficient to show

1. $\sqrt{-g_{44}}\, T = \text{constant}$ (Tolman's Theorem)

2. $\sqrt{-g_{44}}\, G = \text{constant}$ (Klein's Theorem)

3. Thermal equilibrium (No Heat Flow) at all times.

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APPENDIX 1

THE FORM OF THE VISCOUS STRESS TENSOR

We have the result

A1.1 $$S^{\alpha}_{,\alpha} = -\theta_{,\alpha}q^{\alpha} - \frac{1}{T}\epsilon_{\alpha\beta}\tau^{\alpha\beta} \geq 0 \ .$$

Let us assume that q^{α} and $\tau^{\alpha\beta}$ are linear functions of $\theta_{,\alpha}$ and $\epsilon_{\alpha\beta}$. Then

A1.2 $$q^{\mu} = A^{\mu\alpha}\theta_{,\alpha} + B^{\mu\alpha\beta}\epsilon_{\alpha\beta}$$

A1.3 $$\tau^{\mu\nu} = C^{\mu\nu\alpha}\theta_{,\alpha} + D^{\mu\nu\alpha\beta}\epsilon_{\alpha\beta} \ .$$

Let us assume that $\theta_{,\alpha}$ and $\epsilon_{\alpha\beta}$ may be set arbitrarily. Since $q^{\mu}U_{\mu} = \tau^{\mu\nu}U_{\mu} = 0$ we have

A1.4 $$A^{\mu\alpha}U_{\mu} = B^{\mu\alpha\beta}U_{\mu} = C^{\mu\nu\alpha}U_{\mu} = D^{\mu\nu\alpha\beta}U_{\mu} = 0 \ .$$

Take a coordinate system wherein $U^{\mu} \overset{*}{=} (0,0,0,1)$. Then

A1.5 $$\epsilon_{\alpha 4} \overset{*}{=} \epsilon_{4\nu} \overset{*}{=} 0 \quad ; \quad \tau^{\mu 4} \overset{*}{=} \tau^{4\nu} \overset{*}{=} 0 \ .$$

Since $\epsilon_{\alpha\beta}$ is symmetric we may suppose that

A1.6 $$B^{\mu\alpha\beta} = B^{\mu\beta\alpha}$$

A1.7 $$D^{\mu\nu\alpha\beta} = D^{\mu\nu\beta\alpha} \ .$$

We may without loss of generality suppose that

A1.8 $$B^{\mu\alpha 4} \overset{*}{=} B^{\mu 4\alpha} = 0$$

A1.9 $$D^{\mu\nu\alpha 4} \overset{*}{=} D^{\mu\nu 4\alpha} = 0 \quad .$$

Let us now assume that the fluid is isotropic. That is, suppose that $A^{\mu\alpha}$, $B^{\mu\alpha\beta}$, $C^{\mu\nu\alpha}$, $D^{\mu\nu\alpha\beta}$ are invariant under rotations of the spatial coordinates. It is well known (Jeffrey: Cartesian Tensors) that this requires

A1.10
$$\begin{aligned}
A^{ij} &\overset{*}{=} Ag^{ij} \\
B^{ijk} &\overset{*}{=} B\epsilon^{ijk} \\
C^{ijk} &\overset{*}{=} C\epsilon^{ijk} \\
D^{ijk\ell} &\overset{*}{=} D_1 g^{ij} g^{k\ell} + D_2 g^{ik} g^{j\ell} + D_3 g^{i\ell} g^{jk}
\end{aligned}$$

Then equations A1.6 and the skew-symmetry of ϵ^{ijk} imply

$$B \overset{*}{=} 0 \quad .$$

Since A1.4, A1.8 imply

A1.11 $$B^{\alpha\beta\gamma} \overset{*}{=} 0$$

when α, β, or γ is 4 , we have

A1.12 $$B^{\alpha\beta\gamma} \overset{*}{=} 0 \quad .$$

The equation A1.3 and the symmetry of $\tau^{\mu\nu}$ show that

A1.13 $$C \overset{*}{=} 0 \quad .$$

Equation A1.5 shows that

A1.14 $$C^{\mu 4\alpha} \overset{*}{=} C^{4\mu\alpha} \overset{*}{=} 0 \quad .$$

The only non-zero components of $C^{\alpha\beta\gamma}$ are those of the form C^{ij4}. To satisfy isotropy we must have

$$C^{ij4} \overset{*}{=} C_1 g^{ij} \quad .$$

Thus

A1.15 $$C^{\alpha\beta\nu} = C_1 \, \Delta^{\alpha\beta} U^{\nu} \quad .$$

Since $A^{4\alpha} \overset{*}{=} 0$ we have

A1.16 $$A^{\mu\nu} \overset{*}{=} \begin{bmatrix} 1 & 0 & 0 & A^{14} \\ 0 & 1 & 0 & A^{24} \\ 0 & 0 & 1 & A^{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we define a vector A^{μ} by

A1.17 $$A^{\mu} \overset{*}{=} A^{\mu 4}$$

we have the result

A1.18 $$q^{\mu} = A \, \Delta^{\mu\nu} \theta_{,\nu} + A^{\mu} \dot{\theta} \quad .$$

Thus a change in θ with time causes a heat flow in the direction of A^{μ}. This contradicts isotropy unless

A1.19 $$A^{\mu} \overset{*}{=} 0 \quad .$$

We may now write

A1.20 $$A^{\mu\nu} = A\,\Delta^{\mu\nu} \quad .$$

In view of equations A1 9, A1.4, A1.10, we may put

A1.21 $$D^{\mu\nu\alpha\beta} = D_1\,\Delta^{\mu\nu}\,\Delta^{\alpha\beta} + D_2\,\Delta^{\mu\alpha}\,\Delta^{\nu\beta} + D_3\,\Delta^{\mu\beta}\,\Delta^{\nu\alpha} \quad .$$

The forms of q^{α} and $\tau^{\alpha\beta}$ are then

A1.22 $$q^{\mu} = A\,\Delta^{\mu\nu}\theta_{,\nu}$$

A1.23 $$\tau^{\mu\nu} = C\,\Delta^{\mu\nu}\dot{\theta} + D_1\,\Delta^{\mu\nu}\,\Delta^{\alpha\beta}\varepsilon_{\alpha\beta} + (D_2+D_3)\,\Delta^{\mu\alpha}\Delta^{\nu\beta}\epsilon_{\alpha\beta} \quad .$$

Observe that A1.1 implies that

A1.24 $$S^{\alpha}_{,\alpha} = O(\epsilon^2) \quad .$$

Since

A1.25 $$S^{\alpha} = S\rho U^{\alpha} + \frac{1}{T}\,\zeta^{\alpha} - \theta\xi^{\alpha} \quad .$$

We have in view of 2.5.8

A1.26 $$S^{\alpha}_{,\alpha} = S_{,\alpha}\rho U^{\alpha} + S(\rho U^{\alpha})_{,\alpha} + O(\epsilon^2) \quad .$$

Since

A1.27 $$(\rho U^{\alpha} + \xi^{\alpha})_{,\alpha} = 0$$

we have

A1.28 $$(\rho U^{\alpha})_{,\alpha} = O(\epsilon^2) \quad .$$

Thus

A1.29 $$\dot{S} = O(\epsilon^2) \quad .$$

Now

A1.30 $$\dot{\theta} = \left(\frac{\partial\theta}{\partial\rho}\right)_S \dot{\rho} + \left(\frac{\partial\theta}{\partial S}\right)_\rho \dot{S}$$

$$= \left(\frac{\partial\theta}{\partial\rho}\right)_S \dot{\rho} + O(\epsilon^2) \quad .$$

According to A1.28

A1.31 $$\dot{\rho} = -\rho U^{\alpha}_{,\alpha} + O(\epsilon^2) \quad .$$

Thus, to within $O(\epsilon^2)$

A1.32 $$\dot{\theta} = -\left(\frac{\partial\theta}{\partial\rho}\right)_s \rho \; U^{\alpha}_{,\alpha} = -\left(\frac{\partial\theta}{\partial\rho}\right)_s \rho \, \epsilon^{\alpha}_{\alpha} \quad .$$

The form of $\tau^{\mu\nu}$ is then

A1.33 $$\tau^{\mu\nu} = D_1 \Delta^{\mu\nu} \left(1 - \left(\frac{\partial\theta}{\partial\rho}\right)_s \rho\right) \epsilon^{\alpha}_{\alpha}$$

$$+ \left(D_2 + D_3\right) \Delta^{\mu\alpha} \Delta^{\nu\beta} \epsilon_{\alpha\beta} \quad .$$

Substitution of A1.22, A1.33, into A1.1 finally gives us the form of q^{μ} and $\tau^{\mu\nu}$ as

$$q^{\mu} = -K \Delta^{\mu\nu} \theta_{,\nu} \qquad K \geq 0$$

$$\tau^{\mu\nu} = -\lambda_1 \Delta^{\alpha\mu} \Delta^{\beta\nu} \epsilon_{\alpha\beta} - \lambda_2 \Delta^{\mu\nu} \Delta^{\alpha\beta} \epsilon_{\alpha\beta} \quad ;$$

$$\lambda_1, \lambda_2 \geq 0 \quad .$$

APPENDIX 2

REPLACEMENT OF $O(\epsilon^2\rho) + O(\epsilon^2\mu)$ by $O(\epsilon^2 P)$.

We may write the equation of state in the form

A2.1 $$P = P(U,\rho) \quad ; \quad U = (\mu-\rho)c^2 \; .$$

Thus

A2.2 $$\frac{P_*-P}{P} = \left(\frac{\partial P}{\partial U}\right)_\rho \frac{U_*-U}{P} + \left(\frac{\partial P}{\partial \rho}\right)_U \frac{\rho_*-\rho}{P}$$

$$= \left(\frac{\partial P}{\partial U}\right)_\rho \frac{\mu_*-\mu}{P}\, c^2 + \left[\left(\frac{\partial P}{\partial \rho}\right)_U - c^2\left(\frac{\partial P}{\partial U}\right)_\rho\right] \frac{\rho_*-\rho}{P}$$

$$= \left(\frac{\partial P}{\partial U}\right)_\rho O\left(\frac{\epsilon^2\mu c^2}{P}\right) + \left[\frac{1}{c^2}\left(\frac{\partial P}{\partial \rho}\right)_U - \left(\frac{\partial P}{\partial U}\right)_\rho\right] O\left(\frac{\epsilon^2\rho c^2}{P}\right) .$$

Now $\dfrac{P}{\mu c^2} >> \epsilon$, so we may replace $O(\frac{\epsilon^2\mu c^2}{P})$ by $O(\epsilon^2)$. Thus

A2.3 $$\frac{P_*-P}{P} = \left(\frac{\partial P}{\partial U}\right)_\rho O(\epsilon^2) + \left[\frac{1}{c^2}\left(\frac{\partial P}{\partial \rho}\right)_U - \left(\frac{\partial P}{\partial U}\right)_\rho\right] O(\epsilon^2) \; (\frac{\rho}{\mu}) \; .$$

Now $P < \frac{1}{3}\mu c^2$, cf. Taub [4] so we may write

A2.4 $$\frac{P_*-P}{P} = \left(\frac{\partial P}{\partial U}\right)_\rho O(\epsilon^2) + \left[\frac{1}{c^2}\left(\frac{\partial P}{\partial \rho}\right)_U - \left(\frac{\partial P}{\partial U}\right)_\rho\right] O(\epsilon^2)\, \frac{1}{\eta} \; .$$

Under classical conditions $\left(\frac{\partial P}{\partial U}\right)_\rho$ and $\left[\frac{1}{c^2}\left(\frac{\partial P}{\partial P}\right)_U - \left(\frac{\partial P}{\partial U}\right)_\rho\right]$ are $O(1)$.

Under relativistic conditions all substances exist as gases. It has been shown (cf. Israel, Basic topics of the relativistic theory of shock waves) that for Boltzmann, Fermi, and Bose gases

A2.5
$$\left|\left(\frac{\partial P}{\partial U}\right)_\rho\right| \leq \frac{2}{3} \quad ; \quad \left|\frac{1}{c^2}\left(\frac{\partial P}{\partial \rho}\right)_U - \left(\frac{\partial P}{\partial U}\right)_\rho\right| \frac{1}{\eta} < \frac{2}{3} \ .$$

We may conclude that for all "reasonable" equations of state

$$\frac{P_* - P}{P} = O(\epsilon^2)$$

under all conditions.

APPENDIX 3

PROOF THAT $\rho\theta = \sum \rho_A \theta_A$.

Consider a region of volume V_1 , containing fluid at equilibrium. Let

$$\text{A3.1} \quad \left\{ \begin{array}{lcl} U_1 & = & \text{Total energy in the region} \\ S_1 & = & \text{Total entropy in the region} \\ M_A & = & \text{Total mass of fluid A in the region} \\ T & = & \text{Temperature of the region} \\ P & = & \text{Pressure of the region} \end{array} \right. .$$

Define

$$G_1 = U_1 + P_1 V_1 - T_1 S_1 \quad .$$

It is well known (cf. Zemansky, Heat and Thermodynamics; McGraw Hill, pp. 417-419) that

$$\text{A3.2} \qquad dG_1 = SdT + VdP + \sum_{k=1}^{N} \mu_k dM_k$$

and

$$\text{A3.3} \qquad G_1 = \sum_{k=1}^{N} \mu_k M_k \quad .$$

Now our θ is related to G_1 by

A3.4 $$\theta = \frac{G_1}{\rho V_1 T} .$$

Hence

A3.5 $$\theta = \sum_{A=1}^{N} \left(\frac{\mu_A}{T} \right) \left(\frac{\rho_A}{\rho} \right) .$$

Differentiating A3.4 yields, with A3.2 and A3.5

A3.6 $$d\theta = \frac{1}{\rho T} dP + \eta d \left(\frac{1}{T} \right) + \sum_k \frac{\mu_k}{T} d \left(\frac{\rho_k}{\rho} \right) .$$

Since

A3.7 $$\theta = \frac{\mu + P}{\rho T} - S$$

We have

A3.8 $$dS = d \left(\frac{\mu + P}{\rho T} \right) - d\theta$$

$$TdS = d \left(\frac{\mu}{\rho} \right) + Pd \left(\frac{1}{\rho} \right) - \sum_k \mu_k \ d \left(\frac{\rho_k}{\rho} \right)$$

If we then put

A3.9 $$\theta_A = \frac{\mu_A}{T}$$

Equations 3.2.1 and 3.2.5 follow.